1 Lagrange interpolation polynomial

Consider the set of \( u(x_j) \) for the discrete points \( \{x_j\}, \quad j = 0, \ldots, N \). The interpolation polynomial for \( u(x_j) \) in the Lagrange form is

\[
p(x) = \sum_{j=0}^{N} \phi_j(x)u(x_j)
\]

(1)

where \( \phi_j(x) \) is called Lagrange interpolation coefficient and is defined by

\[
\phi_j(x) = \prod_{m=0}^{N} \left( \frac{x - x_m}{x_j - x_m} \right).
\]

(2)

Lagrange interpolation (1) is also defined as follows

\[
\phi_j(x) = \frac{S_N(x)}{S_N'(x_j)(x - x_j)}.
\]

(3)

where \( S_N(x) \) is defined by

\[
S_N(x) = \prod_{j=0}^{N} (x - x_j).
\]

(4)

2 Chebyshev interpolation

Chebyshev polynomial is defined by

\[
T_n(x) = \cos(n\theta), \quad x = \cos \theta.
\]

(5)

For the \( S_N(x) \) in the interpolation polynomial (3), we choose the following form

\[
S_N(x) = (1 - x^2) \frac{dT_N(x)}{dx}.
\]

(6)
By using the Chebyshev polynomial (5), we can expand $S_N(x)$ as follows:

$$S_N(x) = (1 - x^2)\frac{dT_N(x)}{dx} = \sin^2 \theta \frac{-N \sin n\theta}{-\sin \theta} = N \sin \theta \sin N\theta,$$

so that $S_N(x)$ becomes zero for

$$N\theta = \pi j, \quad j : \text{integer}.$$

Therefore we define

$$\theta_j = \frac{\pi j}{N}, \quad j = 0, 1, \cdots, N,$$

then we get

$$S_N(x_j) = 0, \quad x_j = \cos \frac{\pi j}{N}.$$

Here $x_j$ are the Gauss-Chebyshev-Lobatto points. Thus, using (6) and Chebyshev polynomial (5), interpolation coefficient (3) is defined by

$$\phi_j(x) = \frac{(1 - x^2)\frac{dT_N(x)}{dx}}{d_j(x - x_j)}, \quad j = 0, 1, \cdots, N,$$

where $d_j$ are

$$d_j = S'_N(x_j) = -c_j N^2 T_N(x_j), \quad \frac{d}{dx} = \frac{d}{dx}$$

and

$$c_j = 2, \quad \text{for } j = 0, N, \quad c_j = 1 \quad \text{for } 0 < j < N.$$  

Proof) Derivative of $S_N(x)$ with $x$ is

$$S'_N(x) = -N \frac{\cos \theta}{\sin \theta} \sin N\theta - N^2 \cos N\theta$$

Here for $j = 1, \cdots, N - 1$, $\sin \theta_j \neq 0$ and $\sin N\theta_j = 0$, so that we obtain $S'_N(x_j) = -N^2 T_N(x_j)$. While $\sin \theta_j = 0$ and $\sin N \times \theta_j = 0$ for $j = 0, N$. Therefore $S'_N$ includes $0/0$, so that we apply the L'Hopital theorem for the first term of the right hand side

$$S'_N(x_j) = S'_N(\theta = 0 \quad \text{or} \quad \pi) = -N \frac{\cos \theta}{\sin \theta} \sin N\theta - N^2 \cos N\theta \quad \text{L'Hopital}$$

$$\frac{-N}{\cos \theta} \sin N\theta + \frac{N \cos \theta \cos N\theta}{\cos \theta} - N^2 \cos N\theta = -2N^2 T_N(x_j)$$

2
3 Derivation of Chebyshev differentiation matrix

Consider Gauss-Chebyshev-Lobatto points (or Chebyshev points, for short) in the $x \in [-1,1]$ defined by

$$x_j = \cos \theta_j = \frac{\pi j}{N}, \quad j = 0, 1, \cdots, N. \quad (12)$$

Given a grid function $u$ defined on the Chebyshev points, we obtain a discrete derivative $w$ in two steps:

- Let $p$ be the unique polynomial of degree $\leq N$ with $p(x_j) = u_j$, $0 \leq j \leq N$.
- Set $w_j = p'(x_j)$.

This operation is linear, so it can be represented by multiplication by an $(N+1) \times (N+1)$ matrix, which we shall denote by $D_N$:

$$w_i = (D_N)_{ij} \psi_j \quad \text{for} \quad i = 0, 1, \cdots, N. \quad (13)$$

Here $N$ is an arbitrary positive integer, even or odd. And, $(D_N)_{ij}$ represents the $(i, j)$ elements of the matrix $D_N$.

In order to derive the matrix $D_N$, consider the interpolation polynomial (1) and the interpolation coefficients (9). From the derivative $d p(x)/dx = u(x_j) d\phi_j(x)/dx$ we get

$$w_i = (D_N)_{ij} \psi_j.$$  

Considering eqn. (9) we obtain

$$\frac{d}{dx} \phi_j(x) = \frac{1}{(x-x_j)^2} \left\{ \left[-2x \frac{d}{dx} T_N(x) + (1-x^2) \frac{d^2}{dx^2} T_N(x) \right](x-x_j) \right. $$

$$\left. -(1-x^2) \frac{d}{dx} T_N(x) \right\}, \quad j = 0, 1, \cdots, N. \quad (15)$$

(1) Non-diagonal elements : $1 < i < N, \; 1 < j < N, \; i \neq j$

if we put $x = x_i$ in eqn.(15), we obtain

$$(D_N)_{ij} = \frac{1}{d_j x_i - x_j} \frac{1}{1-x_i^2} \frac{d^2}{dx^2} T_N(x_i), \quad (16)$$

where we used the relation $-2x_i dT_N(x_i)/dx = 0$ in the bracket $[\;]$. That is $\theta_i = i\pi/N, \; \sin \theta_i \neq 0$

$$\left. -2x \frac{d}{dx} T_N(x) \right|_{x=x_i} = -2 \cos \theta_i \frac{-N \sin i\pi}{\sin \theta_i} = 0$$

3
Similarly this relation was applied for the second term of the ordinary differential equation of Chebyshev polynomial

\[(1 - x^2) \frac{d^2}{dx^2} T_n(x) - x \frac{d}{dx} T_n(x) + n^2 T_n(x) = 0,\]  

(17)

then we get the following relation:

\[(1 - x_i^2) \frac{d^2}{dx^2} T_N(x_i) = -N^2 T_N(x_i).\]  

(18)

Substituting this into eqn.(16) we obtain

\[(D_N)_{ij} = \frac{d_i}{d_j} \frac{1}{x_i - x_j} = \frac{c_i}{c_j} \frac{T_N(x_i)}{T_N(x_j)} \frac{1}{x_i - x_j}.\]  

(19)

Considering \(T_N(x_i) = \cos[N(i\pi)/N] = \cos i\pi = (-1)^i\) and (11), we get

\[(D_N)_{ij} = \frac{c_i}{c_j} \frac{(-1)^{i+j}}{x_i - x_j}, \quad c_i = \begin{cases} 2 & \text{for } i = 0 \text{ or } N \\ 1 & \text{otherwise} \end{cases}\]  

(20)

(II) Diagonal elements \((i = j)\) \(0 < i, j < N\)

When \(x \to x_j\), the first derivative (15)

\[\frac{d}{dx} \phi_j(x) \bigg|_{x \to x_j} = \frac{1}{(x - x_j)^2} \left\{ \left[ Y(x) \right] (x - x_j) - (1 - x^2) \frac{d}{dx} T_N(x) \right\}_{x \to x_j}\]

becomes \(0/0\) because \((1 - x_j^2)dT_N(x_j)/dx = 0\). Where, we defined \(Y(x) = -2xT_N(x_j)/dx + (1 - x^2)d^2 T_N(x_j)/dx^2\), while \(x \to x_j\), \(Y(x_j) \neq 0\). We can apply the L’Hôpital theorem to the above derivative and get

\[\frac{d}{dx} \phi_j(x) \bigg|_{x \to x_j} = \frac{1}{2(x - x_j)} \left\{ -2xT_N' + (1 - x^2)T_N'' \right\} + (x - x_j) \left[ -2T_N' - 2xT_N'' - 2xT_N''' + (1 - x^2)T_N'' \right] + 2xT_N' - (1 - x^2)T_N'' \]

\[= \frac{1}{2(x - x_j)} \left\{ (x - x_j) \left[ -2T_N' - 4xT_N'' + (1 - x^2)T_N''' \right] \right\} = \frac{1}{2} \left\{ -2T_N' - 4xT_N'' + (1 - x^2)T_N''' \right\},\]  

(21)

so that the diagonal elements of the Chebyshev differentiation matrix are

\[(D_N)_{jj} = -\frac{1}{2d_j} \left\{ 2T_N'(x_j) + 4x_jT_N''(x_j) - (1 - x_j^2)T_N'''(x_j) \right\}.\]  

(22)
Because of $j \neq 0$, $N$, $\sin \theta_j \neq 0$, then $T_N'(x_j) = 0$. And from eqn. (17)

$$-(1 - x^2)T''_N(x) = -3xT'_N(x) + (N^2 - 1)T'_N(x).$$

Considering $T_N'(x_j) = 0$, we substitute this relation into eqn. (22) we obtain

$$(D_N)_{jj} = -\frac{1}{2d_j}x_jT''_N(x_j).$$

Moreover for $T''_N(x_j)$ and making use of (18), the diagonal elements are derived as follows:

$$(D_N)_{jj} = -\frac{-x_jN^2T_N(x_j)}{2d_j(1 - x_j^2)} = \frac{-x_j}{2(1 - x_j^2)}. \quad (23)$$

(III) $i = 0, j = 0$ or $i = N, j = N$
Because $x_j = 1$ for $j = 0$ and $x_j = -1$ for $j = N$, then the last term of the right hand side of (22) vanishes. And we know

$$T_N(1) = 1, \quad T_N(-1) = (-1)^N, \quad T'_N(1) = N^2, \quad T'_N(-1) = (-1)^N N^2$$

and also

$$T''_N(1) = (\frac{N^2 - 1)N^2}{3}, \quad T''_N(-1) = (-1)^N (\frac{N^2 - 1)N^2}{3}.$$ Therfore the numerator and denominator of eqn.(22) are

$$i = 0, j = 0: \quad 2T'_N(1) + 4T''_N(1) = \frac{2N^2(2N^2 + 1)}{3}, \quad 2d_0 = -4N^2T_N(1)$$

$$i = N, j = N: \quad 2T'_N(-1) + 4T''_N(-1) = (-1)^N \frac{2N^2(2N^2 + 1)}{3},$$

$$2d_0 = -4N^2T_N(-1),$$

so that finally we get

$$(D_N)_{00} = \frac{2N^2 + 1}{6}, \quad (D_N)_{NN} = -\frac{2N^2 + 1}{6}. \quad (24)$$

Derivation of $T''_N(\pm 1)$:

Ordinary differentiation equation for Chebyshev polynomial in (V) we used

$$-(1 - x^2)T''_N(x) = -3xT'_N(x) + (N^2 - 1)T'_N(x).$$

By using for $x = 1$ in above equation, we obtain

$$T''_N(1) = \frac{(N^2 - 1)}{3} T'_N(1) = \frac{(N^2 - 1)N^2}{3}.$$ For $T''_N(-1)$, we derive similarly.

5
References

