Permeability of a Densely Packed Bed of Solids with Modified Brinkman equation.

Shigenobu Itoh

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Abstract

The permeability of a porous medium is discussed here. The porous mediums, for example a granular bed and a fibrous mass, are assumed to be random arrays of identical rigid-spheres and rods. A statistical formulation developed by Lundgren (1972) is applied to these systems. Then an obtained equation is, what we call, Brinkman equation. The correlation between the permeability and the volume fraction,c, of the solids are found for a granular bed and a fibrous mass. Here the relation between the volume concentration ,c and a porosity, ϵ, is as follows: $c = 1 - \epsilon$.

When a conditional average is considered around a test sphere, a mean number of the spheres in contact with a certain point decreases as the point approaches to the test sphere. This particular region is called an intermediate layer. Then, in order to get a more correct permeability, it must be calculated by taking into account of the intermediate layer. The property of the layer is represented by a Darcy resistance with a coefficient which depends on the distance between the point and the center of the test sphere as well as on c; when the layer is not taking into account, it depends only on c. Then the generated conditional averaged equation for the flow might be called the modified Brinkman equation. Therefore the basic equations in this article are the continuity equation and the modified Brinkman equation. However, it is difficult to solve the basic equations due to a non-constant coefficient of Darcy resistance in the modified Brinkman equation. Then, the Galerkin method is used to obtain an approximate solution of the basic equations. With those obtained approximate solutions, the permeability is obtained by using the self-consistency condition: Darcy resistance should be equal to the total drag on the spheres contained in unit volume of the material. The obtained permeability of the bed of spheres agrees with the experimental data almost all the entire volume concentration, especially near $c = 2/3$. That is, **the main result of this article is shown in Figure 3.1**.

In the case of the bundle of cylindrical rods, two typical flows are considered: in the first, the flow is assumed to be parallel to the axes of rods; in the second, the flow is perpendicular to the axes. In each case, the intermediate layer is considered. Then we found that the permeability for the transverse flow through the random array of rods is smaller than that through spheres for c >0.5.

March 30, 1985

Preface to the revised edition

26 years have passed since this paper was published. During this time the

computer has undergone a remarkable development. Now, we get very powerful and inexpensive Personal Computer (PC for brevity) and spreadsheet software (Microsoft Excel) that runs on it. I adopted the method called the Galerkin method to solve the fundamental equations in this article approximately, and PC was used for the calculation. It is now possible to easily search for more accurate approximate solutions with the powerful PC and the spreadsheet software. For the case of the bed of spheres and the case of the longitudinal flow along cylinders, those are the same as the results of 26 years ago, however we had to correct the approximate solution of the transverse flow around cylinders.

Sections 3.3 and 4.4 are newly added to this revised edition, where parametric studies of trial function have been conducted. Moreover Appendix E is also newly added to the latest edition, in which the velocity profile around the test sphere is shown for each volume concentration of sphere.

May 4, after the Tohoku-Pacific Ocean Earthquake, 2011

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(Please replace underscore A and (_) with @ and dot (.), respectively.)

The paper on the permeability of the bed of spheres is quoted in the following books and papers at the present time.

1) P. M. Adler, A. Nadim and H. Brenner; "Rheological models of suspensions" - Advances in chemical engineering, 1990 - Elsevier
"In the same vein, **Itoh(1983)** employed Eq.(5.5) [$\mathcal{F} = \mu\alpha^2 < \mathbf{U} >$] to calculate α using the same self-consistent argument, but with the inclusion of an intermediate fluid layer around the test sphere. Theoretical results compared well with experimental results."

2) A. J. C. Ladd, "Hydrodynamic transport coefficients of random dispersions of hard spheres", - The Journal of chemical physics, 1990
" The transport properties of a random dispersion of hard spheres has been a subject of theoretical interest for a long time (Happel and Brenner,1986). Properties of particular interest are:
• The permeability of a fixed array of spheres (Brinkman,1947; Tam, 1969; **Itoh, 1983**; Kim and Russel,1985),
• The single-particle and collective mobility coefficients (sedimentation velocity) $\cdots$
• The viscosity $\cdots$"

3) Donald A. Drew and Stephen L. Passman, "Theory of multicomponent fluids", Springer-Verlag Berlin Heidelberg 1997

4) S. Haussener, I. Jerjen, P. Wyss and A. Steinfeld, "Tomography-based determination of effective heat and mass transport properties of complex multi-phase media", Transactions of the ASME, Journal of Heat Transfer, JANUARY 2012, Vol. 134
"**Itoh(1983)** determined the permeability of a random array of rigid spheres accounting for the so called intermediate layer, a particular state formed around a test sphere."
5) Mark F. Hurwitz, "Drag in a Porous Medium: An Example of the Use of Ensemble Averaged Hydrodynamic Potentials" , 89 - 97, 1998
, etc.

Contents

Chapter 1

Introduction

1.1 Phenomena of an infiltration and Darcy's law

Phenomena of the flow through a granular bed (or in a porous medium) play an important role in nature and in industrial apparatuses. We can see the infiltration of a fluid through porous media in many fields such as petroleum engineering, soil mechanics, ground water hydrology, and sanitary engineering, etc: oil flows observed when oil is pumped out of an oil reservoir, ground water flows in an upper layer of the earth and its flows to wells. Also in earth sciences, flows through a soil have an important bearing on various agricultural problems, e.g., the drainage of water supplied by natural and irrigation sources, etc.

On the other hand, an application of the infiltration is seen in chemi-

Figure 1.1: A tower for absorption or desorption.

cal engineering. For generating chemical reaction, many industrial systems involve towers packed with special shapes or crushed solids through which fluids of raw material flow, and chemically react upon each other. Such towers serve as contracting device to bring together gases and liquids for the purpose of absorption or desorption. (cf., Fig.1.1)

Darcy's law is widely employed for investigating those flows through a granular bed or a porous medium. The law expresses that the rate of flow is proportional to pressure drop through a bed of fine particles. The coefficient is called the permeability of a granular bed. The permeability is, in general, a tensorial quantity. However, if the medium is homogeneous and isotropic, it becomes scalar. For such a medium, Darcy's law is expressed as

$$\frac{Q}{A} = -\frac{k}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{1.1}$$

where k is the intrinsic permeability, μ the viscosity of the fluid, $\mathrm{d}p/\mathrm{d}x$ the pressure gradient along the flow path, and Q denotes the flow rate through a cross-section having an area of A. The original expression of Darcy's law is seen in Appendix A. From now on, we abbreviate the intrinsic permeability to the permeability.

On the other hand, by careful experiments on laminar flow through a pipe Hagen (1839) and Poiseuille (1840) established the following relationship:

$$\frac{Q}{A} = -\frac{d^2}{32\mu}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{1.2}$$

where d and A denote the diameter and the cross-section of the pipe, respectively, $\mathrm{d}p/\mathrm{d}x$ is the pressure gradient along it, and Q denotes the rate of flow.

1.2 Various studies for predicting the permeability

The permeability k is a measure of the 'ease' with which a fluid passes through the porous material. It is macroscopic property of the medium and characterizes the transport phenomena[1],[2]. For many years, various attempts[3],[4] have been made to find the correlation between the permeability and the parameter of a porous medium such as the fractional pore space (porosity) or the volume fraction of the solids (volume concentration), the grain size distribution, packing and orientation of constituent grains. Among those correlation's, the relation between permeability and porosity is the most important one. Then in order to establish the correlation theoretically, we represent the porous media by theoretical models. Those models can be treated mathematically. A capillaric model is the simplest way to find the correlation, and an approach by 'drag theory', mentioned below, is another way to find it.

(a) capillaric model

With a straight capillaric model, a porous medium is supposed to be a bundle of straight capillaries entering the medium on one face and emerging on the opposite face with 'average' pore diameter $\bar{d}$. If there are n such capillaries per unit area of cross-section of the model, the flow per unit area, q, is given by the law of Hagen-Poiseuille:

$$q = -\frac{n\pi\bar{d}^4}{128\mu}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{1.3}$$

where μ is as usual the viscosity and $\mathrm{d}p/\mathrm{d}x$ is the pressure gradient along the capillary. As the flow can also be expressed by Darcy's law (1.1) it follows that $k = \epsilon d^2/32$, where ϵ is the porosity and we used $\epsilon = n\pi d^2/4$. It is clear, however, that a model consisting of parallel capillaries gives permeability in one direction only, and that, it is known that this result does not correctly represent the connection between permeability and porosity in porous media as it is actually observed.

In straight and parallel type models each capillary is supposed to go through a granular bed without variation of diameter. This is far from reality. For a more realistic one, a tortuous capillaric model is supposed because the path of a stream line through the pore space will be tortuous. An average length of the flow path, L_E, is greater than the length of the porous medium L. However, this model introduce an additional parameter (the 'tortuosity' $T = L_E/L$) into the model. The introduction of an unknown parameter T is not very satisfactory.

(b) drag theory of permeability

This approach was initiated by Emersleben(1924). In this theory, the solid part of the porous material is treated as obstacles to flow of the viscous fluid. The drag of the fluid on each portion of the solid is estimated from Navier-Stokes equations, and the sum of all drags is thought to be equal to the resistance of the porous medium to flow.

Brinkman(1947) thought the shape of obstacles as spheres. For a homogeneous and isotropic porous medium, he considered a slow flow through the medium as a flow through a dense swarm of particles. Then he thought that the force on a particle situated in a swarm of particles could be calculated as if it were a solid particle embedded in a porous material. He represented the porous material by modified Stokes's equation:

$$-\nabla p + \mu\nabla^2\mathbf{U} - \frac{\mu}{k}\mathbf{U} = \mathbf{0} \tag{1.4}$$

where p is the pressure of fluid, $\mathbf{U}$ the apparent velocity of the fluid, and the last term on the left hand side of eq.(1.4) is called Darcy resistance (cf., eq.(1.1). In this thesis we call eq.(1.4) Brinkman equation. The permeability obtained by him agreed with experimental data for the sparse distribution of spheres. However, Brinkman's theory has been received some skepticism because Darcy's equation is derived empirically. Furthermore, when $c = 2/3$, the permeability becomes zero (in other words, the fluid cannot permeate the swarm of particles, though there still remains a pore space). About twenty years later, Tam[5] derived the Brinkman equation theoretically by using an ensemble average.

Another approach to obtain the drag on a sphere is called 'cell technique' used Brenner[3]. In this theory, a granular bed is supposed to be divided into a number of identical unit cells. Each cell contains one particle (a constituent grain).

A typical cell is assumed to be a sphere, and its radius must be specified such that the porosity of unit cell is identical to that of the granular bed. The fluid in a cell is governed by the creeping motion equations. In this unit cell technique, the following assumption is important: the entire disturbance due to each particle is confined to the cell with which it is associated. This model, however, contains an arbitrariness of the choice of the boundary conditions on the cell, so that it is difficult to employ the most reasonable boundary condition.

1.3 Permeability of a granular bed by an ensemble average

The permeability is a bulk property of the medium. To obtain such a macroscopic quantity of the medium, it may be necessary to employ some kind of averaging. An ensemble-averaging approach has been developed to study the fluid-particle system[6]-[11]. In this theory, a porous medium is usually assumed to be an assemblage of spheres which are fixed at random in space by homogeneous and isotropic manner. And fluids flow very slowly through it. We show an ensemble-average formulation by Lundgren[12] in Chapter2. As is mentioned in the drag theory, it becomes important to calculate the drag on a sphere. A mean drag of flow exerted on a (test) sphere can be obtained by using an ensemble average[12]-[18]. Now we review some important works on these problems below.

Tam[5] derived the Brinkman equation theoretically by treating a collection of particles as point force in Stokes flow and by ensemble averaging over all particles positions except that the primary (or a test) particle. By inserting the test particle in the mean flow and by calculating the average flow field around it, he obtained a drag formula for the particle. The formula had the same difficulty as Brinkman's when $c = 2/3$. We show the same calculation as Tam's in the section3.1.

Howells[14] studied the flow through a random array of spheres under the condition that the volume concentration of the spheres is small ($c \ll 1$). He developed a point-force technique which satisfied the equation of continuity and the Brinkman equation. Then, he represented presence of each sphere by a distribution of the point forces on it. When a second sphere is fixed, a mean drag on a test sphere is calculated, including two-sphere interaction which is expanded in powers of c, provided $c \ll 1$. One significant contribution to the drag is Brinkman's result and another is due to the principal excluded-volume effect; he calculated the effect by using the method analogous to Faxen's formula[9]. The result is much the same as Brinkman's in region $c \ll 1$.

Lundgren[12] studied the permeability of a moderately dense collection of spheres. He advanced a statistical formulation due to Saffman[19], and derived a modified Stokes' equation with an unknown additional force term. Then, he assumed that an explicit expression of the additional force is to be $\bar{A} < \mathbf{U} > +\bar{B}\nabla^2 < \mathbf{U} >$, where $\bar{A}$ and $\bar{B}$ are unknown constants to be determined and $< \mathbf{U} >$ is an ensemble average velocity. From an ensemble of spheres we single out the sub-ensemble for which one sphere (i.e., a test sphere) has the same position in each member. Then, an average over all the sub-ensemble yields a conditional average.(cf., Fig.1.2)

When the conditional-average flow past the test sphere is considered, a void region of sphere-centers occurs around the sphere, because the test

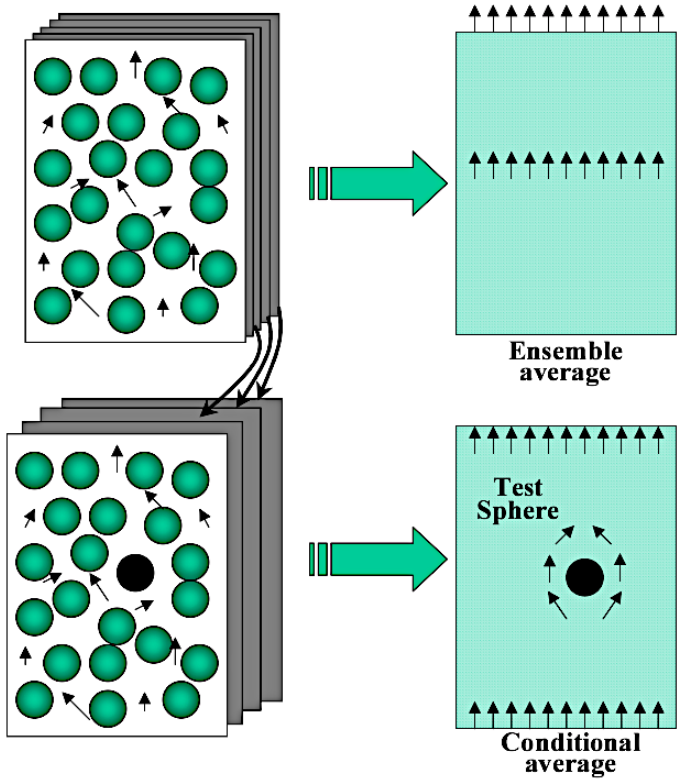

Figure 1.2: An ensemble average and a conditional average.

sphere excludes all spheres which overlap it[20]. For spheres of radius a, the radius of the void region is $2a$. But, for simplifying his calculation Lundgren neglected the void region and he introduced an unknown effective viscosity instead. The resulting permeability is an unreasonable one which increases rapidly when $c > 0.3$. He noted that in order to get a valid result at higher concentration, it would be necessary to take account of the void sphere around each particle.

Buevich and Marcov[16] refined Lundgren's work in some sense. They also noted the test sphere is surrounded by a particular region in which a mean number of the spheres in contact with a certain point around the test sphere decreases as the point approaches to the sphere. They called the region an intermediate layer, but neglected the layer in their calculation. Their result contains the same difficulty as Lundgren's.

The volume concentration of the solids of a granular bed would be large in general. So, in order to predict a valid permeability of a densely packed bed it would be necessary to take the void region or the intermediate layer into account. The mean flow around the test sphere is governed by Brinkman equation, and it includes the Darcy resistance which is linearly proportional to the mean velocity. Then, for a homogeneous and isotropic medium, we assume that the intermediate layer is represented by imposing the following condition: a coefficient in the resistance term depends on the distance between the point and the center of the test sphere as well as on c; when the layer is neglected, it depends only on c. For the homogeneous and isotropic medium, we discuss the correlation between the permeability and the volume concentration by considering the layer in section3.2.

Here, we refer briefly to the study by Neal and Nader[21]. They adopted a cell model and used Brinkman equation derived statistically by Lundgren. A result obtained by them agreed with experiment throughout the entire concentration range. They proposed the drag on a sphere in the porous media as $F_{\mathrm{sph}} = 6\pi\mu\mathbf{U}\xi(\alpha,\beta)$, where μ is a fluid viscosity, and $\mathbf{U}$, its velocity, α and β dimensionless parameters. However the derivation of their main result, $\xi(\alpha,\beta)$, is not so clear[1].

Finally we must note an ensemble-average approach to the porous-medium problem by Prager[6]. It is different from 'drag theory'. He estimated the resistance of a porous medium to a fluid streaming through it by minimizing the rate of energy dissipation. Then, he obtained valid lower bounds on the resistance. (cf., Oshima[22])

[1] cf., Happel and Brenner[3] 4-22 Concentric Spheres

1.4 Permeability of fibrous material

We are also concerned with the infiltration of fluid through a fibrous material such as cloth, felt, filter paper, etc. The first study of this problem was done by Emersleben(1925). He thought the fibrous material as a bundle of fibers. The study by Emersleben is seen in Scheidegger[4]. In this case, two typical flows are considered: in the first, flow is assumed to be parallel to the axes of cylindrical fibers (longitudinal flow); in the second, flow is perpendicular to the axes (transverse flow). See Fig.1.3.

In this thesis we assume the fibrous material is a random but homogeneous assemblage of cylindrical rods of the same radius. Then, we treat the permeability of the mass by taking the intermediate layer into account in Chapter4.

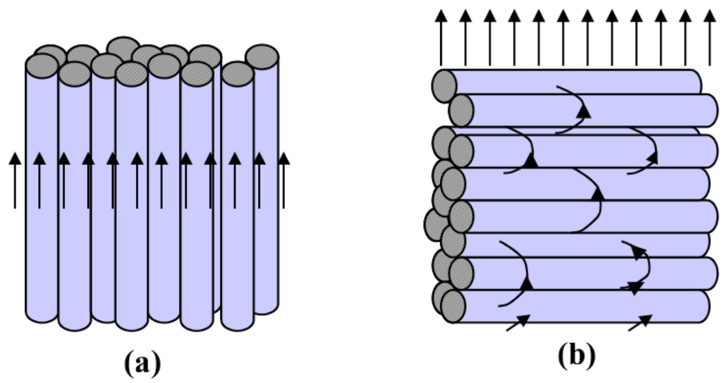

Figure 1.3: Two-typical flows:(a) longitudinal flow; (b) transverse flow.

1.5 "Fluid and particles" -related problems; turblence, suspension, etc.,

As is mentioned above, we treat a very slow flow through the assemblage of particles in this thesis. Therefore, inertial effects can be neglected and the fluid flowing through it is governed by the creeping motion equations. However, in the case of a high seepage velocity inertial effects must be taken into account because the distortion of streamlines occurs in a porous medium, owing to change in direction of motion of fluid particles. Furthermore, as the seepage velocity becomes considerably higher a turbulent motion will occur in it. It is difficult to study these problems thoroughly, and works on those are fewer in number at present.

In this thesis, our purpose is to discuss the permeability of densely packed beds of solids. On the other hand, however, the low solids content system (dilute) have been studied for many years. In this case, particles may move relative to each other, as well as with respect to the fluid: such phenomena are called as 'suspension'[8]; particularly, if the fluid experiences no net motion, the particle movement is designated as 'sedimentation'[9]. At intermediate concentration of particles, the particles are not held immobile by the interparticle contacts, due to the fluid motion: in such a case, 'phenomena of fluidization' will occur. We are also interested in these subjects, however, they are not treated here.

Chapter 2

Statistical Formulation

2.1 Ensemble average

We consider the slow flow through a random array of rigidly fixed solid spheres. The array is a collection of N identical solid spheres of radius a. It is assumed that those spheres are distributed in a statistically homogeneous and isotropic manner. Field variables (e.g., velocity, pressure, etc.) at a point $\mathbf{r}$ in the fluid-sphere system are determined by the (instantaneous) positions of the centers of N spheres. A large number of realizations with the same macroscopic boundary conditions make up an ensemble, and an average over the values of some quantity occurring in these realizations is an ensemble average.

A probability density function for the location of center of the N spheres being at $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N$ is denoted by $P_N(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N)$. The normalization condition is

$$\int_{D^N} P_N(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N) \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 \mathrm{d}\mathbf{r}_3 \cdots \mathrm{d}\mathbf{r}_N = 1 \tag{2.1}$$

where D^N is the N-time Cartesian product of D. A probability density function for one-sphere is denoted by

$$P_1(\mathbf{r}_1) = \int_{D^{N-1}} P_N(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N) \mathrm{d}\mathbf{r}_2 \mathrm{d}\mathbf{r}_3 \cdots \mathrm{d}\mathbf{r}_N. \tag{2.2}$$

We also introduce a conditional probability density function $P_{N-1}(\mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N | \mathbf{r}_1)$, which is for the location of the center of the spheres being at $\mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N$ when the center of a sphere is fixed at $\mathbf{r}_1$. The following relationship is well known:

$$P_{N-1}(\mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N | \mathbf{r}_1) = \frac{P_N(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)}{P_1(\mathbf{r}_1)}. \tag{2.3}$$

Similarly, if the position of two spheres are fixed, the conditional probability density function for the remaining $N-2$ spheres is given by the formula

$$P_{N-2}(\mathbf{r}_3, \mathbf{r}_4, \cdots, \mathbf{r}_N | \mathbf{r}_1, \mathbf{r}_2) = \frac{P_N(\mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N)}{P_2(\mathbf{r}_1, \mathbf{r}_2)}, \tag{2.4}$$

where P_2 is the two-sphere probability density function defined by the same way as in (2.2).

Now we express an average of some quantity $G(\mathbf{r})$ in the system as follows:

$$< G(\mathbf{r}) >= \int_{D^N} G(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) P_N(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) \mathrm{d}\mathbf{r}_2 \cdots \mathrm{d}\mathbf{r}_N. \tag{2.5}$$

where $G(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)$ is the quantity at $\mathbf{r}$ when the spheres are located at $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$. From now on the ensemble average is denoted by $< \cdot >$. A number density function, $n(\mathbf{r})$, of spheres is defined by making use of Dirac delta function:

$$n(\mathbf{r}) = n(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) = \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i)$$

Then, by using (2.2) one obtains the averaged number density

$$< n(\mathbf{r}) >= N P_1(\mathbf{r}). \tag{2.6}$$

A conditional average of $G(\mathbf{r})$ is defined by

$$< G(\mathbf{r}|\mathbf{r}_1) >= \int_{D^{N-1}} G(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) P_{N-1}(\mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N | \mathbf{r}_1) \mathrm{d}\mathbf{r}_2 \mathrm{d}\mathbf{r}_2 \cdots \mathrm{d}\mathbf{r}_N. \tag{2.7}$$

Finally it should be noted that the ensemble averaging and the operations of differentiation with respect to $\mathbf{r} = (x_1, x_2, x_3)$ commute for continuous functions, that is,

$$\begin{aligned}
\left\langle \frac{\partial G}{\partial x_i} \right\rangle &= \int_{D^N} \frac{\partial G(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)}{\partial x_i} P_N(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 \cdots \mathrm{d}\mathbf{r}_N \\
&= \frac{\partial}{\partial x_i} \int G(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) P_N(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 \cdots \mathrm{d}\mathbf{r}_N \\
&= \frac{\partial}{\partial x_i} < G(\mathbf{r}) > . \qquad (2.8)
\end{aligned}$$

2.2 Ensenble averaged Equations

In this section we derive an ensemble averaged field equations for the fluid-sphere system. Since the velocity around each sphere is very small in our problem, the fluid is supposed incompressible and inertia term in the equation of motion can safely be neglected. Then, for each member of an ensemble the velocity and the pressure fields of fluid is governed by the equation of continuity and Stokes' equation (or the creeping motion equations):

$$\nabla \cdot \mathbf{u}(\mathbf{r}) = 0, \tag{2.9}$$

$$-\nabla p(\mathbf{r}) + \mu \nabla^2 \mathbf{u}(\mathbf{r}) = \mathbf{0}, \tag{2.10}$$

where $\mathbf{u}, p$ and μ denote the velocity, pressure and viscosity of fluid. No-slip condition is imposed on the surface of the spheres. Inside the sphere we define

$$\mathbf{u}(\mathbf{r}) = \mathbf{0} \cdots\cdots \text{and} \cdots\cdots \nabla p(\mathbf{r}) = \mathbf{0}. \tag{2.11}$$

It is convenient to introduce a function $H(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)$ which specifies the fluid region in one realization:

$$H(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) = \begin{cases} 1 \cdots\cdots\cdots \mathbf{r} \in D_{\text{Fluid}}, \\ 0 \cdots\cdots\cdots \mathbf{r} \in D_{\text{Solid}}, \end{cases} \tag{2.12}$$

where D_{Fluid} means that the point $\mathbf{r}$ is in the fluid, and also D_{Solid} means $\mathbf{r}$ in the Solid. The function $H(\mathbf{r})$ is given explicitly by

$$H(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) = 1 - \sum_{i=1}^{N} \mathcal{H}(a - |\mathbf{r} - \mathbf{r}_i|), \tag{2.13}$$

where $\mathcal{H}$ is the Heaviside step function defined by

$$\mathcal{H}(x) = \begin{cases} 0 \cdots\cdots\cdots x < 0, \\ 1 \cdots\cdots\cdots x \geq 0. \end{cases}$$

With this expression for $H(\mathbf{r})$ we get

$$\begin{aligned} < H > &= \int H(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) P_N(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 \cdots \mathrm{d}\mathbf{r}_N \\ &= 1 - \int \sum_{i=1}^{N} \mathcal{H}(a - |(\mathbf{r} - \mathbf{r}_i|) P_N(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 \cdots \mathrm{d}\mathbf{r}_N \\ &= 1 - N \int \mathcal{H}(a - |\mathbf{r} - \mathbf{r}_1|) P_1(\mathbf{r}_1) \mathrm{d}\mathbf{r}_1 \\ &= 1 - \int_{\|\mathbf{r}-\mathbf{r}_1\| \leq a} N P_1(\mathbf{r}_1) \mathrm{d}\mathbf{r}_1 = 1 - \int_{\|\mathbf{r}-\mathbf{r}_1\| \leq a} < n(\mathbf{r}_1) > \mathrm{d}\mathbf{r}_1 \\ &= 1 - c, \end{aligned} \tag{2.14}$$

where eqs.(2.2) and (2.6) have been used, and c is the volume concentration of spheres. In (2.14) $< n(\mathbf{r}_1) >$ is assumed to be uniform because of the homogeneity of the system.

An ensemble average of the velocity $\mathbf{u}(\mathbf{r})$ is defined by

$$< H\mathbf{u}(\mathbf{r}) >= \int H(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)\mathbf{u}(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)\mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2\cdots\mathrm{d}\mathbf{r}_N, \tag{2.15}$$

where $\mathbf{u}(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)$ is the velocity at $\mathbf{r}$ when the spheres are located at $\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N$. From now on the ensemble averaged velocity $< H(\mathbf{u}(\mathbf{r}) >$ is simply denoted by $< \mathbf{U} >$. An ensemble average pressure $< Hp(\mathrm{r}) >$ is defined by the same way as in (2.15).

The Stokes' stress tensor,

$$T_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.16}$$

is defined only in the fluid. By using H an average stress becomes

$$< HT_{ij} >= - < Hp > \delta_{ij} + \mu\left\langle H\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\rangle. \tag{2.17}$$

Because $\partial u_i/\partial x_j$ vanishes inside the sphere, we find

$$\begin{aligned}\left\langle\frac{\partial u_i}{\partial x_j}\right\rangle &= \left\langle(1-H+H)\frac{\partial u_i}{\partial x_j}\right\rangle = \left\langle(1-H)\frac{\partial u_i}{\partial x_j}\right\rangle + \left\langle H\frac{\partial u_i}{\partial x_j}\right\rangle \\ &= \left\langle H\frac{\partial u_i}{\partial x_j}\right\rangle. \end{aligned} \tag{2.18}$$

Then we obtain

$$\left\langle H\frac{\partial u_i}{\partial x_j}\right\rangle = \left\langle\frac{\partial u_i}{\partial x_j}\right\rangle = \frac{\partial}{\partial x_j} < u_i >= \frac{\partial}{\partial x_j} < Hu_i >, \tag{2.19}$$

where we used eq.(2.8). Therefore, by using the above expression we get the averaged Stokes' stress tensor,

$$< HT_{ij} >= - < Hp > \delta_{ij} + \mu\left(\frac{\partial < U_i >}{\partial x_j} + \frac{\partial < U_j >}{\partial x_i}\right). \tag{2.20}$$

We make use of these results as follows. From eqs.(2.18) and (2.19) and the equation of continuity, we find

$$\nabla\cdot < \mathbf{U}(\mathbf{r}) >= 0. \tag{2.21}$$

On the other hand, Stokes' equation is rewritten by using Stokes stress tensor (2.16) as $\mathbf{0} = \mathrm{div}\mathbf{T}$:

$$\begin{aligned}(\mathrm{div}\mathbf{T})_i &= \frac{\partial}{\partial x_j}T_{ij} = \frac{\partial}{\partial x_j}\left\{-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\} \\ &= -\frac{\partial}{\partial x_i}p + \mu\frac{\partial^2}{\partial x_j \partial x_j}u_i + \mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) \\ &= -\frac{\partial}{\partial x_i}p + \mu\frac{\partial^2}{\partial x_j \partial x_j}u_i = 0,\end{aligned}$$

where the repeated index implies summation over all values of that index, and the equation of continuity has been used. Then, an ensemble average of Stokes' equation yields

$$\mathbf{0} =< H\mathrm{div}\mathbf{T} >= \mathrm{div} < H\mathbf{T} > - < \mathbf{T}(\mathbf{r}) \cdot \nabla H > . \tag{2.22}$$

where the last term on the right is the contribution from those realizations of the ensemble for which the point is on a solid boundary. Further, by using eq.(2.20) we get

$$-\nabla < Hp(\mathbf{r}) > +\mu\nabla^2 < \mathbf{U}(\mathbf{r}) > - < \mathbf{T}(\mathbf{r}) \cdot \nabla H >= \mathbf{0}. \tag{2.23}$$

The last term $< \mathbf{T}\cdot\nabla H >$ in the above equation can be put in more explicit form by using

$$\nabla H(\mathbf{r}) = \nabla\left\{1 - \sum_{i=1}^{N}\delta(a - \|\mathbf{r} - \mathbf{r}_i\|)\right\} = \sum_{i=1}^{N}\frac{\mathbf{r} - \mathbf{r}_i}{\|\mathbf{r} - \mathbf{r}_i\|}\delta(\|\mathbf{r} - \mathbf{r}_i\| - a). \tag{2.24}$$

Then

$$\begin{aligned}&< \mathbf{T}\cdot\nabla H >= \int_{D^N}\mathbf{T}\cdot\nabla H P_N \mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2\cdots\mathrm{d}\mathbf{r}_N \\ &= \int_{D^N}\sum_{i=1}^{N}\mathbf{T}\cdot\frac{\mathbf{r} - \mathbf{r}_i}{\|\mathbf{r} - \mathbf{r}_i\|}\delta(\|\mathbf{r} - \mathbf{r}_i\| - a)P_N\mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2\cdots\mathrm{d}\mathbf{r}_N \\ &= \int_{D^N}\mathbf{T}\cdot\frac{\mathbf{r} - \mathbf{r}_1}{\|\mathbf{r} - \mathbf{r}_1\|}\delta(\cdot)P_N\mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2\cdots\mathrm{d}\mathbf{r}_N + \cdots \\ &\qquad + \cdots\cdots\cdots + \int_{D^N}\mathbf{T}\cdot\frac{\mathbf{r} - \mathbf{r}_N}{\|\mathbf{r} - \mathbf{r}_N\|}\delta(\cdot)P_N\mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2\cdots\mathrm{d}\mathbf{r}_N \\ &= N\int_D \delta(\|\mathbf{r} - \mathbf{r}_1\| - a)\frac{\mathbf{r} - \mathbf{r}_1}{\|\mathbf{r} - \mathbf{r}_1\|}P_1(\mathbf{r}_1)\cdot\left\{\int_{D^{N-1}}\mathbf{T}P_{N-1}\mathrm{d}\mathbf{r}_2\mathrm{d}\cdots\mathrm{d}\mathbf{r}_N\right\}\mathrm{d}\mathbf{r}_1 \\ &= \int_D < n(\mathbf{r}_1) >< \mathbf{T}(\mathbf{r}|\mathbf{r}_1) > \cdot\frac{\mathbf{r} - \mathbf{r}_1}{\|\mathbf{r} - \mathbf{r}_1\|}\delta(\|\mathbf{r} - \mathbf{r}_1\| - a)\mathrm{d}\mathbf{r}_1 \\ &=< n(\mathbf{r}) > \int_D < \mathbf{T}(\mathbf{r}|\mathbf{r}_1) > \cdot\mathbf{n}_1\delta(\|\mathbf{r} - \mathbf{r}_1\| - a)\mathrm{d}\mathbf{r}_1,\end{aligned} \tag{2.25}$$

where $\mathbf{n}_1 = (\mathbf{r}-\mathbf{r}_1)/\|\mathbf{r}-\mathbf{r}_1\|$ and eqs.(2.3) and (2.6) have been used. Because of the delta function, the volume integral in eq.(2.25) can be converted to a surface integral:

$$\vec{\mathcal{F}}(\mathbf{r}) =< \mathbf{T} \cdot \nabla H >= \int_{S_1} < n >< \mathbf{T}(\mathbf{r}|\mathbf{r}_1) > .\mathbf{n}_1 \mathrm{d}S, \tag{2.26}$$

where S_1 is a surface of a sphere of radius a at $\mathbf{r}_1$. From now on, we call the force $\vec{\mathcal{F}}$ an additional force.

An equation for $< \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >$ can be obtained by the above procedure, using a conditional probability in place of P_N. The conditional averaged equations around the (test) sphere are

$$\nabla\cdot < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >= 0, \tag{2.27}$$

$$-\nabla < Hp(\mathbf{r}|\mathbf{r}_1) > +\mu\nabla^2 < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) > - < \mathbf{T}(\mathbf{r}|\mathbf{r}_1) \cdot \nabla H >= \mathbf{0}, \tag{2.28}$$

where $\mathbf{T}(\mathbf{r}|\mathbf{r}_1)$ is the stress at $\mathbf{r}$ when the test sphere is fixed at $\mathbf{r}_1$. The last term $< \mathbf{T}(\mathbf{r}|\mathbf{r}_1) \cdot \nabla H >$ on the left hand side in eq.(2.28) is put in more explicit form:

$$\begin{aligned} &< \mathbf{T}(\mathbf{r}|\mathbf{r}_1) \cdot \nabla H >= N\int_D \delta(\|\mathbf{r}-\mathbf{r}_2\| - a)\frac{\mathbf{r}-\mathbf{r}_2}{\|\mathbf{r}-\mathbf{r}_2\|} P_1(\mathbf{r}_2|\mathbf{r}_1) \\ &\times\left\{\int_{D^{N-2}} \mathbf{T}(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N) P_{N-2}(\mathbf{r}_3,\cdots,\mathbf{r}_N|\mathbf{r}_1,\mathbf{r}_2)\mathrm{d}\mathbf{r}_3\cdots\mathrm{d}\mathbf{r}_N\right\}\mathrm{d}\mathbf{r}_2 \\ &= < n(\mathbf{r}|\mathbf{r}_1) > \int_{S_2} < \mathbf{T}(\mathbf{r}|\mathbf{r}_1,\mathbf{r}_2) > \cdot\mathbf{n}_2\mathrm{d}S = \vec{\mathcal{F}}(\mathbf{r}|\mathbf{r}_1), \end{aligned} \tag{2.29}$$

where $\mathbf{n}_2 = (\mathbf{r}-\mathbf{r}_2)/\|\mathbf{r}-\mathbf{r}_2\|, < n(\mathbf{r}|\mathbf{r}_1) >$ is the averaged number density around the test sphere, and $\mathbf{T}(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2)$ is the stress tensor at $\mathbf{r}$ when the spheres are fixed at $\mathbf{r}_1$ and $\mathbf{r}_2$. This differs from eq.(2.25), because of the non-uniform density field $< n(\mathbf{r}|\mathbf{r}_1 >$ caused by the presence of spheres at $\mathbf{r}_1$. The stress $< \mathbf{T}(\mathbf{r}|\mathbf{r}_1,\mathbf{r}_2) >$ in eq.(2.29) depends on $< \mathbf{U}(\mathbf{r}|\mathbf{r}_1,\mathbf{r}_2) >$ which is the average velocity field when the positions of two spheres are fixed, and the average velocity $< \mathbf{U}(\mathbf{r}|\mathbf{r}_1,\mathbf{r}_2) >$ is unknown. We can continue this procedure to derive an equation for $< \mathbf{U}(\mathbf{r}|\mathbf{r}_1,\mathbf{r}_2) >$. However, it would have a resistance term which depends on the average velocity of $< \mathbf{U}(\mathbf{r}|\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) >$, when three spheres are fixed, and so on. This leads to a hierarchy of equations. Hence, a certain assumption is needed to truncate the hierarchy of them and is made in the next section.

2.3 Assumption of the additional forces $\mathcal{F}$ and the existence of the intermediate layer

In the case of a slow flow, we can represent a dimension of the additional forces $\mathcal{F}$ as follows:

$$|\mathcal{F}| \approx \mu U/L^2 \tag{2.30}$$

where L denotes the characteristic length, U the velocity and μ the viscosity.

According to the article by Lundgren[12] and Buevich & Marcov[16], $\mathcal{F}(\mathbf{r})$ and $\mathcal{F}(\mathbf{r}|\mathbf{r}_1)$ are assumed to be linear functions of $< \mathbf{U}(\mathbf{r}) >$ and $< \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >$, respectively. Then, in the light of (2.30) we propose a most simple form of the functional $\mathcal{F}[< \mathbf{U} >]$:

$$\mathcal{F} = \mu\{\alpha(c)\}^2 < \mathbf{U}(\mathbf{r}) > . \tag{2.31}$$

Here, α is the unknown quantity which depends on c and has a dimension L^{-1}. (A quantity α^{-1} is called the shielding radius.)

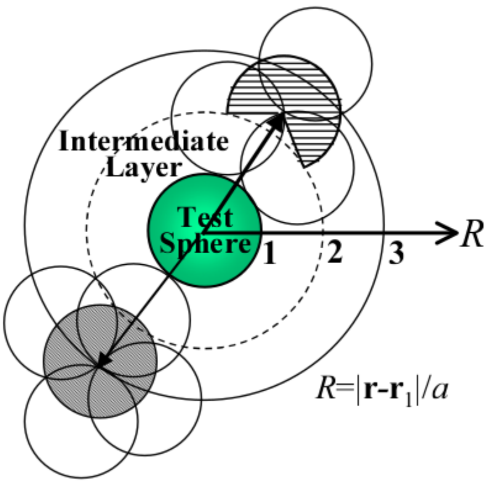

Figure 2.1: Intermediate layer around the test sphere.

Now we must explain the following fact before we discuss a functional form $\mathcal{F}(\mathbf{r}|\mathbf{r}_1)$. As is mentioned in § 2.1, the presence of the test sphere at $\mathbf{r}_1$ imposes the constraints on the field around it. (See Fig.2.1.) In the region $\|\mathbf{r} - \mathbf{r}_1\| \geq 3a$ the test sphere doesn't influence on the possible position of other spheres which are in contact with a point $\mathbf{r}$. The locus of centers of those spheres yields a sphere of radius a, centered at $\mathbf{r}$. On the other hand, in region $a \leq \|\mathbf{r} - \mathbf{r}_1\| < 3a$ the test sphere has influence on the location of sphere-centers. Then the centers of the spheres in contact with $\mathbf{r}$ do not form a complete sphere centered at $\mathbf{r}$. The spherical surface area depends on $\|\mathbf{r}-\mathbf{r}_1\|$. We call this region the intermediate layer and show it also in Fig.2.1. Therefore, in region $\|\mathbf{r}-\mathbf{r}_1\| \geq 3a$ a coefficient of $< \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >$ in additional force term $\mathcal{F}(\mathbf{r}|\mathbf{r}_1)$ is the same as that of $< \mathbf{U}(\mathbf{r}) >$, whereas in the region $a \leq \|\mathbf{r}-\mathbf{r}_1\| < 3a$ the coefficient depends on $\|\mathbf{r}-\mathbf{r}_1\|$, in addition to the dependence the volume fraction c. Thus we make an assumption of $\mathcal{F}(\mathbf{r}|\mathbf{r}_1)$:

$$\mathcal{F}(\mathbf{r}|\mathbf{r}_1) = \begin{cases} \mu\alpha^2 Q_S(\|\mathbf{r}-\mathbf{r}_1\|) < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >, & a \leq \|\mathbf{r}-\mathbf{r}_1\| < 3a, \\ \mu\alpha^2 < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >, & \|\mathbf{r}-\mathbf{r}_1\| \geq 3a, \end{cases} \tag{2.32}$$

where the function Q_S represents the dependence on the radial distance of the intermediate layer for the spheres. A difference between the two regions around the test sphere mainly consists in the mean number of spheres which are in contact with the point $\mathbf{r}$. Hence we propose that the difference can be represented by following Q_S which is a proportion of the spherical area centered at $\mathbf{r}$ to $4\pi a^2$. Then, considering the location of the spheres surrounding the test sphere, we obtain (under a condition that the spheres are impenetrable and the array is homogeneous and isotropic)

$$Q_S(R) = \frac{1}{2}\left(1 + \frac{R^2-3}{2R}\right), \tag{2.33}$$

where $R = \|\mathbf{r}-\mathbf{r}_1\|/a$.

In the previous section, the problem about the additional force took place. In other word, we didn't have a clue to truncate the hierarchy of the fundamental equations. However, in this section we determined a form of the additional force explicitly: the additional force is represented by the averaged quantity which is the same level average of other term in the ensemble averaged differential equation. By this scheme, the hierarchy of the equations has been truncated. Moreover, by the conditional average when the test sphere is fixed, a special region occurs around the test sphere. We call it the intermediate layer. With this layer, we are able to represent the size of the test sphere which is neglected by Brinkman and Lundgren.

2.4 Self-consistency condition

In the previous section the infinite hierarchy of the equations has been closed by the assumptions of the form of the additional force term. However, an unknown quantity $\alpha(c)$ was introduced as a result of the assumption. But, the unknown quantity α can be obtained by carrying out following process. The stress $< H\mathbf{T}(\mathbf{r}|\mathbf{r}_1) >$ is expressed by $< \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >$ and $< Hp(\mathbf{r}|\mathbf{r}_1) >$ which are the solutions of the fundamental equations (2.27) and (2.28). By integrating the stress on the surface of the sphere we obtain a drag $\mathbf{f}(\mathbf{r}_1)$ on it. Because of the homogeneity of the array the drag $\mathbf{f}(\mathbf{r}_1)$ is independent of the point $\mathbf{r}_1$. Then, since n stands for the mean number density of the spheres the total drag exerted on the spheres in unit volume of the material can be expressed by $n\mathbf{f}(\mathbf{r})$. On the other hand, the additional force, $\mathcal{F}(\mathbf{r})$, corresponds to a mean resistance per unit volume. So we require $\mathcal{F}(\mathbf{r})$ should be equal to $n\mathbf{f}$:

$$n\mathbf{f} = \mu\alpha^2 < \mathbf{U} > . \tag{2.34}$$

Thus α can be determined directly by the requirement, which is called the self-consistency condition.

An alternative way of the determination of α is considered in this paragraph. Because of the homogeneity of the system, the stress $< H\mathbf{T}(\mathbf{r}|\mathbf{r}_1) >$ is independent of the sphere-center at $\mathbf{r}_1$. Therefore we can recognize that the following relation:

$$\int_{S_1} < H\mathbf{T}(\mathbf{r}|\mathbf{r}_1) > .\mathbf{n}_1 \mathrm{d}S = \int_S < H\mathbf{T}(\mathbf{r}|\mathbf{r}_1) > .\mathbf{n}\mathrm{d}S = \mathbf{f}(\mathbf{r}_1),$$

where S_1 is the locus of the center of the spheres which are in contact with the point $\mathbf{r}$, and S is the surface of the test spheres. (cf., Fig.2.2)

$$\begin{aligned}
\mathcal{F}(\mathbf{r}) &= n\int_{S_1} < H\mathbf{T}(\mathbf{r}|\mathbf{r}_1) > .\mathbf{n}_1 \mathrm{d}S = n\int_S < H\mathbf{T}(\mathbf{r}|\mathbf{r}_1) > .\mathbf{n}\mathrm{d}S \\
&= n\mathbf{f}(\mathbf{r}) = \mu\alpha^2 < \mathbf{U} >,
\end{aligned}$$

which is the same condition as (2.34).

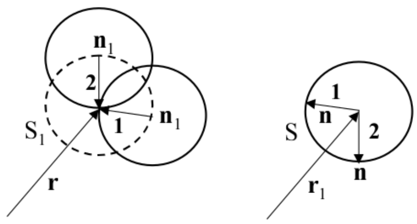

Figure 2.2: S_1:Locus of the centers of spheres, S:the surface of the sphere.

2.5 Correspondence of an ensemble average quantity to an experiment

Our purpose is to predict the permeability and to compare it with experiment. Hence we must refer to a physical meaning of the quantities appearing in the basic equations (2.21) and (2.23). In an experiment on the permeability we observe a volume averaged velocity, $\mathbf{U}$. An integration of $\mathbf{U}$ over an element of surface yields the average volume of the fluid flowing across the surface per unit time, and it corresponds to $\mathbf{Q}$ in eq.(1.1). Therefore, we shall take for granted that the usual ergodicity property of equality of the ensemble average and a volume (or an integral) average in statistically homogeneous system. (cf., Batchelor[8] § 3) Then, the ensemble averaged velocity $< \mathbf{U} >$ is equal to the volume averaged velocity $\bar{\mathbf{U}}$, so that the mean velocity $< \mathbf{U} >$ can be observed.

On the other hand, we can not observe the ensemble averaged pressure $< Hp(\mathbf{r}) >$, so we shall introduce

$$\bar{p}(\mathbf{r}) = \frac{< Hp(\mathbf{r}) >}{< H >} = \frac{< Hp(\mathbf{r}) >}{(1-c)}. \tag{2.35}$$

The mean pressure $\bar{p}(\mathbf{r})$ is the average only on the realizations of the ensemble for which the point is in the fluid. The mean pressure $\bar{p}$ is a physically measurable quantity[1].

By using the observable quantities $< \mathbf{U}(\mathbf{r}) >$ and $\bar{p}(\mathbf{r})$ we can rewrite eq.(2.23) in the following form:

$$-\nabla(1-c)\bar{p}(\mathbf{r}) + \mu\nabla^2 < \mathbf{U}(\mathbf{r}) > -\mu\alpha^2 < \mathbf{U}(\mathbf{r}) >= \mathbf{0}. \tag{2.36}$$

Thus if $< \mathbf{U}(\mathbf{r}) >$ is constant $\mathbf{U}_0$, eq.(2.36) becomes

$$-\nabla\bar{p}(\mathbf{r}) - \mu\frac{\alpha^2\mathbf{U}_0}{1-c} = \mathbf{0}. \tag{2.37}$$

An analogous equation to eq.(2.37) is seen in De Wiest[23]. Remembering eq.(1.1) and replacing Q/A by U_0, one obtains the permeability, k:

$$k = \frac{1-c}{\alpha^2} = \frac{(1-c)a^2}{B^2}, \tag{2.38}$$

where B is a non-dimensional quantity because α has a dimension L^{-1}. Thus our problem is reduced to a determination of values of B.

Here we summarize this chapter. The unconditional and the conditional flow fields are governed by the following equations: the equations of continuity and motion and the averaged Stokes' stress tensor are written.

[1] It might be called the mean interstitial pressure.

For the unconditional flow field, that is Brinkman equation expressed by $(1-c)\bar{p}$,

$$\nabla. < \mathbf{U}(\mathbf{r}) >= 0, \tag{2.39}$$

$$-\nabla(1-c)\bar{p}(\mathbf{r}) + \mu\nabla^2 < \mathbf{U}(\mathbf{r}) > -\mu\alpha^2 < \mathbf{U}(\mathbf{r}) >= \mathbf{0}, \tag{2.40}$$

$$< \mathbf{T}(\mathbf{r}) >= -(1-c)\bar{p}\mathbf{I} + \mu(\nabla < \mathbf{U}(\mathbf{r}) > +\tilde{\nabla} < \mathbf{U}(\mathbf{r}) >), \tag{2.41}$$

where $\mathbf{I}$ is the unit dyadic corresponds to δ_{ij}, and $\nabla \tilde{<} \mathbf{U}\mathbf{r}) >$ means the transpose of $\nabla < \mathbf{U}\mathbf{r}) >$.

For the conditional flow field, that is modified Brinkman equation is

$$\nabla. < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >= 0, \tag{2.42}$$

$$-\nabla < Hp(\mathbf{r}|\mathbf{r}_1) > +\mu\nabla^2 < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) > -\mu\alpha^2 Q_S < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >= \mathbf{0}, \tag{2.43}$$

$$< \mathbf{T}(\mathbf{r}|\mathbf{r}_1) >= -(1-c) < Hp(\mathbf{r}|\mathbf{r}_1) > \mathbf{I} + \mu(\nabla < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) > +\nabla < \mathbf{U}\tilde{(}\mathbf{r}|\mathbf{r}_1) >). \tag{2.44}$$

Chapter 3

Permeability of a Random Array of Spheres

3.1 Permeability: the intermediate layer is neglected

In this subsection we shall calculate the permeability by neglecting the intermediate layer, that is, we set $Q_S(r) = 1$ identically. Therefore a calculation here is essentially the same as Brinkman's work. The flow field around the test sphere at $\mathbf{r}_1$ is governed by

$$\begin{cases} \nabla . < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >= 0, \\ -\nabla < Hp(\mathbf{r}|\mathbf{r}_1) > +\mu\nabla^2 < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) > -\mu\alpha^2 < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >= \mathbf{0}, \end{cases} \tag{3.1}$$

It is convenient to choose a coordinate frame whose origin coincides with the center of the test sphere. And the boundary condition used here are as follows:

$$< \mathbf{U}(\mathbf{r}|\mathbf{O}) >= \mathbf{0}, \qquad \text{on the test sphere,} \tag{3.2}$$

$$< \mathbf{U}(\mathbf{r}|\mathbf{O}) > \longrightarrow \mathbf{U}_0(\text{const.}), \qquad \text{as} \quad \mathbf{r} \to \infty. \tag{3.3}$$

A proposal for solution of eqs.(3.1) exploited by Felderhof[24]-[27] is used here. We choose $\mathbf{U}_0$ to be in the z-direction, so that in the spherical coordinate (r, θ, ϕ) we have

$$\mathbf{U}_0 = U_0\mathbf{e}_z = U_0(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta), \tag{3.4}$$

with unit vectors $\mathbf{e}_z, \mathbf{e}_r$ and $\mathbf{e}_\theta$. Now we introduce the dimensionless radial distance $R = r/a$. A velocity field $< \mathbf{U}(\mathbf{R}|\mathbf{O}) >$ with the same angular dependence as eq.(3.4) is supposed:

$$< \mathbf{U}(\mathbf{R}|\mathbf{O}) >= U_0 \left\{ \phi(R)\cos\theta\mathbf{e}_r - \left[\phi(R) + \frac{R}{2}\dot{\phi}(R)\right]\sin\theta\mathbf{e}_\theta \right\}, \tag{3.5}$$

where a dot indicates the differentiation with respect to R. Equation (3.5) satisfies the equation of continuity. Next, the angular dependence of the pressure must be given by the scalar product $(\mathbf{R}, \mathbf{U}_0)$. Hence we suppose

$$< Hp(\mathbf{R}|\mathbf{O}) >= -\mu U_0 \frac{B^2}{a} \chi(R) \cos\theta. \tag{3.6}$$

Then, by substitution of eqs.(3.5) and (3.6) into eq.(3.1) we obtain a set of coupled equations governing $\phi(R)$ and $\chi(R)$:

$$\ddot{\phi} + \frac{4}{R}\dot{\phi} + B^2\dot{\chi} - B^2\phi = 0, \tag{3.7}$$

$$\ddot{\chi} + \frac{2}{R}\dot{\chi} - \frac{2}{R^2}\chi = 0. \tag{3.8}$$

Similarly, the boundary conditions (3.2) and (3.3) are rewritten in terms of ϕ:

$$\phi(1) = \dot{\phi}(1) = 0 \qquad \text{on} \quad \text{the} \quad \text{test} \quad \text{sphere}, \tag{3.9}$$

$$\phi \longrightarrow 1 \qquad \text{as} \quad R \longrightarrow \infty. \tag{3.10}$$

We shall have a solution of eq.(3.8) immediately:

$$\chi(R) = C_0 R + \frac{C_1}{R^2}, \tag{3.11}$$

where C_0 and C_1 are unknown constants. Substituting eq.(3.11) into eq.(3.7), we obtain

$$\phi(R) = C_0 - \frac{2C_1}{R^3} + \frac{2C_2}{R^3}(1 + BR)\mathrm{e}^{-BR}, \tag{3.12}$$

where C_2 is an unknown constant. The constant C_0 is determined from the condition (3.10), C_1 and C_2, from (3.9):

$$C_0 = 1, \quad C_1 = (3 + 3B + B^2)/2B^2, \quad C_2 = 3\mathrm{e}^B/2B^2. \tag{3.13}$$

The conditional averaged stress tensor (2.44) can be written in terms of ϕ and χ:

$$< H\mathbf{T}(\mathbf{R}|\mathbf{O}) >= \frac{\mu U_0}{a}\Big\{ 2\dot{\phi}\cos\theta \mathbf{e}_R\mathbf{e}_R - \frac{1}{2}(R\ddot{\phi} + 2\dot{\phi})\sin\theta(\mathbf{e}_R\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_R)$$
$$-\dot{\phi}\cos\theta\mathbf{e}_\theta\mathbf{e}_\theta + B^2\chi\cos\theta\mathbf{I}\Big\} \tag{3.14}$$

Integrating the stress on the test sphere we obtain the drag $\mathbf{f}$

$$\mathbf{f} = \mathbf{f}_{\text{Pressure}} + \mathbf{f}_{\text{Friction}} = \frac{4}{3}\pi\mu a \mathbf{U}_0(B^2\chi + \ddot{\phi}) = \frac{4}{3}\pi\mu\mathbf{U}_0 B^2 3C_1. \tag{3.15}$$

Putting C_1, given by eq.(3.13), in eq.(3.15) we get the force $\mathbf{f}$ exerted on the sphere. Finally the self-consistency condition (2.34) is used to determine the B (or α), then we find a quadratic equation for B. It is readily solved and we have

$$B = \frac{9c + 3\sqrt{8c - 3c^2}}{4 - 6c}, \tag{3.16}$$

where we used $c = 4\pi a^3 n/3$. By using eqs.(2.38) and (3.16) we find the formula for the permeability k:

$$k = \frac{(1-c)(3c+4-3\sqrt{8c-3c^2})a^2}{18c}. \tag{3.17}$$

This formula is different from Brinkman's one only by a factor $(1-c)$.

3.2 Permeability: the intermediate layer is introduced

In this section a calculation of permeability with considering the layer is carried out[28]. We also choose the same coordinate frame and use the same notations as in the previous section. The conditional averaged flow field is governed by

$$\begin{cases} \nabla. < \mathbf{U}(\mathbf{R}|\mathbf{O}) >= 0, \\ -\nabla < Hp(\mathbf{R}|\mathbf{O}) > +\mu\nabla^2 < \mathbf{U}(\mathbf{R}|\mathbf{O}) > -\mu\alpha^2 Q_S(R) < \mathbf{U}(\mathbf{R}|\mathbf{O}) >= \mathbf{0}, \end{cases} \tag{3.18}$$

where the definition of Q_S in eqs.(2.32) and (2.33) is used. We assume the same form of the solutions as in eqs.(3.5) and (3.6):

$$< \mathbf{U}(\mathbf{R}|\mathbf{O}) >= U_0 \left\{ \Phi(R)\cos\theta \mathbf{e}_r - \left[\Phi(R) + \frac{R}{2}\dot{\Phi}(R)\right]\sin\theta \mathbf{e}_\theta \right\}, \tag{3.19}$$

$$< Hp(\mathbf{R}|\mathbf{O}) >= -\mu U_0 \frac{B^2}{a}\Pi(R)\cos\theta. \tag{3.20}$$

By substituting eqs.(3.19) and (3.20) into the second equation of (3.18), we have a set of coupled equations

$$\ddot{\Phi} + \frac{4}{R}\dot{\Phi} + B^2\dot{\Pi} - B^2 Q_{\mathrm{S}}\Phi = 0, \tag{3.21}$$

$$\ddot{\Pi} + \frac{2}{R}\dot{\Pi} - \frac{2}{R^2}\Pi - \dot{Q}_{\mathrm{S}}\Phi = 0. \tag{3.22}$$

The boundary conditions are

$$\Phi(1) = \dot{\Phi}(1) = 0 \qquad \text{on the test sphere}, \tag{3.23}$$

$$\Phi \longrightarrow 1 \qquad \text{as} \quad R \longrightarrow \infty. \tag{3.24}$$

It is difficult to solve the coupled equations (3.21) and (3.22), so we shall adopt the direct method in the theory of the differential equation to obtain approximate solutions. To carry out the method a trial function Φ_t is assumed to be

$$\begin{aligned} \Phi_t = \quad & \underbrace{1 - \frac{5}{2R^3} + \frac{3}{2R^5}}_{\text{satisfyB.C.at} \quad R=1 \quad \text{and} \quad \infty} \quad + a_1 \underbrace{R^2} \quad \underbrace{(R-1)^2}_{\Phi=\dot{\Phi}=0 \quad \text{at} \quad R=1} \quad \overbrace{\mathrm{e}^{-2(R-1)}}^{\text{becomes} \quad 0, \quad R\to\infty} \\ & + a_2 R^2 (R-1)^2 \left[\mathrm{e}^{-6(R-1)} + \frac{1}{10}\mathrm{e}^{-3(R-1)}\right] \end{aligned} \tag{3.25}$$

The trial function (3.25) identically satisfies the boundary conditions (3.23) and (3.24). A factor R^2 appearing in the a_j-term is multiplied to cancel the denominator of $\dot{Q}_S$: this cancellation makes the differential equation (3.22) easier to solve.

For $R \geq 3$ $\dot{Q}_S = 0$, so we can readily find the solution of equation (3.22) :

$$\Pi_+(R) = R + \frac{\Sigma}{R^2}, \tag{3.26}$$

where Σ is the unknown constant and the boundary condition (3.24) has been used. For $1 \leq R < 3$, by substituting the trial function eq.(3.25) into eq.(3.22) we get[1]

$$\begin{aligned}
\Pi_-(R) &= \Xi R + \frac{S}{R^2} + R\int_1^R \left(-\frac{\tilde{R}^2}{3}\right)\frac{1}{\tilde{R}^2}(-\dot{Q}_{\mathrm{S}}\Phi_t)\mathrm{d}\tilde{R} \\
&\quad + \frac{1}{R^2}\int_1^R \left(-\frac{\tilde{R}^2}{3}\right)\tilde{R}(\dot{Q}_{\mathrm{S}}\Phi_t)\mathrm{d}\tilde{R}, \\
&= \Xi R + \frac{S}{R^2} + \frac{1}{16}\left(R^2 - 6 + \frac{5}{R} + \frac{5}{R^2} - \frac{6}{R^3} + \frac{1}{R^5}\right) \\
&\quad + a_1\left[\left(\frac{R^4}{2} + \frac{R^3}{2} + \frac{23R^2}{4} + \frac{21R}{2} + \frac{67}{4} + \frac{67}{4R} + \frac{67}{8R^2}\right)\frac{\exp[-2(R-1)]}{8} \right. \\
&\qquad \left. + \frac{5R}{24} - \frac{1483}{192R^2}\right] \\
&\quad + a_2\left\{\left(\frac{R^4}{2} - \frac{R^3}{6} + \frac{55R^2}{36} - \frac{77R}{54} + \frac{82}{6^3} + \frac{164}{6^4R} + \frac{164}{6^5R^2}\right)\frac{\exp[-6(R-1)]}{72} \right. \\
&\qquad + \frac{16R}{3888} - \frac{4894}{6^7R^2} \\
&\quad + \frac{1}{10}\left[\left(R^4 + \frac{4R^3}{3} + \frac{50R^2}{9} + \frac{98R}{27} + \frac{145}{27} + \frac{290}{81R} + \frac{290}{3^5R^2}\right)\frac{\exp[-3(R-1)]}{36} \right. \\
&\qquad \left.\left. + \frac{11R}{3^5} - \frac{1415}{3^7R^2}\right]\right\},
\end{aligned} \tag{3.27}$$

where Ξ and S are unknown constants. The constants S will be determined by the self-consistency condition. The drag $\mathbf{f}$ on the sphere is obtained by

$$\mathbf{f} = \mathbf{f}_{\mathrm{P}} + \mathbf{f}_{\mathrm{F}} = \frac{4}{3}\pi\mu a\mathbf{U}_0[B^2\Pi_-(1) + \ddot{\Phi}_t(1)] = \frac{4}{3}\pi\mu a\mathbf{U}_0 B^2 3S, \tag{3.28}$$

[cf., eq.(3.15)]. Then we find by using eq.(2.34):

$$n\frac{4}{3}\pi\mu a\mathbf{U}_0 B^2 3S = \frac{\mu B^2\mathbf{U}_0}{a^2}, \tag{3.29}$$

[1] $y'' + P(x)y' + Q(x)y = R(x)$, whose special solutions are assumed to be u_1 and u_2. We denote Wronskian by $W(u_1, u_2)$. Then the solution of the differential equation is $y = c_1u_1 + c_2u_2 + u_1\int\frac{-Ru_2}{W(u_1,u_2)}\mathrm{d}x + u_2\int\frac{Ru_1}{W(u_1,u_2)}\mathrm{d}x$. In this case $W(u_1, u_2) = -3/R^3$.

and we get

$$S = \frac{1}{3c} \tag{3.30}$$

On the other hand, Ξ and Σ are to be determined by continuity conditions for the stress tensor at $R = 3$. One of conditions can be obtained from a continuity of $R - R$ component of the stress $< H\mathbf{T}(\mathbf{R}|\mathbf{O}) >$

$$\Pi_-(3) = \Pi_+(3), \tag{3.31}$$

where we use $\Phi_t(R)$ proposed in eq.(3.25). All other components of the stress should also be continuous at $R = 3$. Then, considering eq.(3.21) we find the condition:

$$\dot{\Pi}_-(3) = \dot{\Pi}_+(3). \tag{3.32}$$

Thus Ξ and Σ can be determined by (3.31) and (3.32) such that

$$\begin{aligned} \Xi &= 1 - \frac{131}{729} - 0.1221a_1 - 0.008062a_2, \\ \Sigma &= \frac{1}{3c} - \frac{55}{27} - 1.5318a_1 - 0.05506a_2. \end{aligned} \tag{3.33}$$

There are no unknown constants except B in eq.(3.21) and a_j. To find the solution of eq.(3.21) is equivalent to seeking the stationary function Φ of $J[\Phi]$[30]:

$$J[\Phi] = \int_1^R (R^4\dot{\Phi}^2 + B^2R^4Q_s\Phi^2 - 2B^2R^4\dot{\Pi}_\pm\Phi)\mathrm{d}R. \tag{3.34}$$

Then we regard Φ_t as an expansion of Φ with a_j. The coefficient a_j will be determined by the condition that the functional $J[\Phi]$ is to be stationary: $\partial J[\Phi]/\partial a_j = 0$. For simplifying our calculation we shall use the Galerkin method[31], then eq.(3.34) turns out to be the following equation:

$$\int_1^R \left[-\frac{\mathrm{d}}{\mathrm{d}R}(R^4\dot{\Phi}_\mathrm{t}^2) + B^2R^4Q_s\Phi_\mathrm{t} - B^2R^4\dot{\Pi}_\pm\right]\omega_j\mathrm{d}R = 0, \tag{3.35}$$

or

$$\int_1^R L_1(\Phi_\mathrm{t})\omega_j\mathrm{d}R = 0, \tag{3.36}$$

where $\omega_1 = R^2(R-1)^2\mathrm{e}^{-2(R-1)}$ and $\omega_2 = R^2(R-1)^2[\mathrm{e}^{-6(R-1)}+\mathrm{e}^{-3(R-1)}/10]$, and $L_1(\Phi)$ denotes the differential equation (3.21). By the use of eq.(3.36) we can determine a_j. However, the integration of those are so lengthy that we calculate numerically by using Simpson's 1/3 rule. Though a region of the radial coordinate R is $[1, \infty)$, it is necessary to restrict the region to

$[1, R_F]$ in order to carry out the calculation: R_F is some finite value of R. We set $R_F = 100$, and after the computation we find

$$\begin{cases} a_1 = \left[\left(0.2075 - \dfrac{0.01714}{c}\right) B^4 + \left(0.9702 - \dfrac{0.1043}{c}\right) B^2 + 0.002619\right] /\Lambda, \\ \\ a_2 = \left[\left(10.91 - \dfrac{2.545}{c}\right) B^4 + \left(21.74 - \dfrac{2.226}{c}\right) B^2 + 6.929\right] /\Lambda, \\ \\ \text{where} \\ \Lambda = 0.5520B^4 + 2.699B^2 + 1.600. \end{cases} \tag{3.37}$$

With making use of eqs.(3.25), (3.37) and (3.27), we get $\ddot{\Phi}(1)$ and $\Pi_-(1)$ respectively, then we find the force on the test sphere. Thus the self-consistency condition for determination of B becomes

$$\mathbf{U}_0 \mu c[\Pi_-(1) + \ddot{\Phi}_t(1)] = \mu B^2 \mathbf{U}_0.$$

The above equation is written explicitly by using the solutions:

$$c\left[\left(\Xi + \frac{1}{3c}\right) + (15 + 2a_1 + 2.2a_2)\right] = B^2,$$

then we obtain

$$\left(c\Xi - \frac{2}{3}\right) B^2 + (15 + 2a_1 + 2.2a_2)c = 0. \tag{3.38}$$

Putting Ξ and a_1 and a_2, given by eqs.(3.33) and (3.37) respectively, in eq.(3.38) we find the equation of degree six for B after a cancellation of the denominator of a_j and by rearrangement:

$$B^6 + \frac{6.893 - 32.43c}{M} B^4 + \frac{5.728 - 87.16c}{M} B^2 - \frac{37.87c}{M} = 0,$$

where

$$M = 0.3454 - 0.3395c. \tag{3.39}$$

Root of the algebraic equation is extracted numerically for each value of c and is shown in Table 3.1. For each value of c, a dimensionless quantity γ defined by $\gamma = a/\sqrt{k}$ is shown also in Table 3.1. That is, by considering eq.(2.38)

$$\gamma = \frac{a}{\sqrt{k}} = \frac{B}{\sqrt{1-c}} \tag{3.40}$$

We can see a comparison of the permeability obtained here with the experimental data in Table 3.2. A comparison of γ with other experimental data is shown in Fig.3.1.

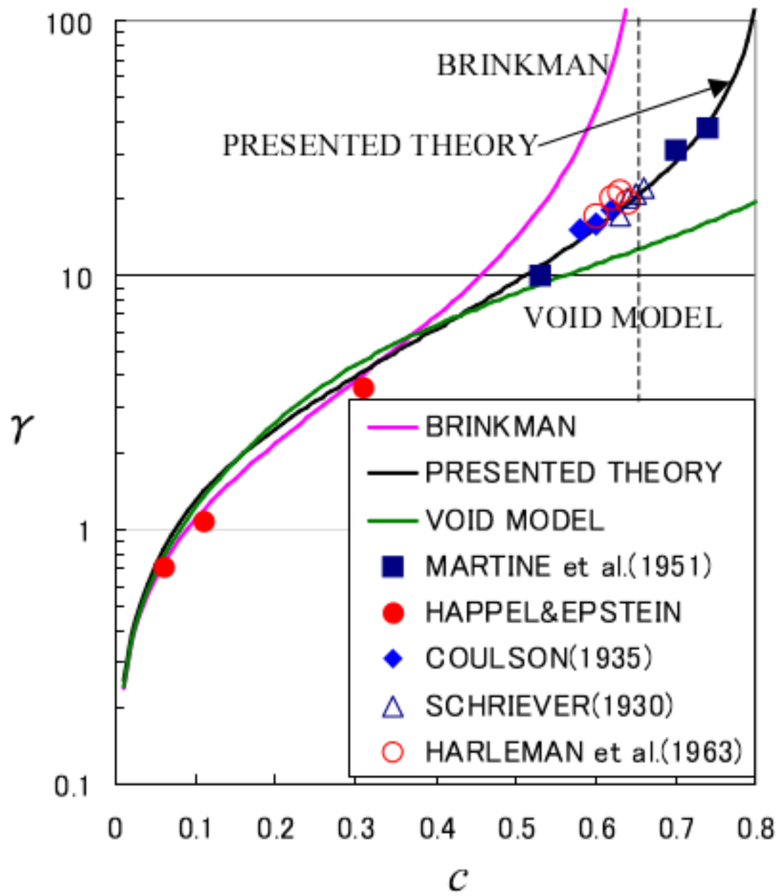

Figure 3.1: Comparison of γ obtained here with experimental data. BRINKMAN: obtained by eq.(3.17). VOID MODEL: obtained from (B.10).

Table 3.1: Values of B and γ

Volume concentration c	B	γ
0.0001	0.0257	0.0257
0.001	0.0816	0.0816
0.01	0.2667	0.2680
0.05	0.6912	0.7091
0.1	1.181	1.245
0.2	2.477	2.769
0.3	4.099	4.899
0.4	5.766	7.444
0.5	7.543	10.67
0.6	9.605	15.19
0.7	12.25	22.37
0.8	16.17	36.16

Table 3.2: Comparison of the permeability obtained here with experimental data [Harleman et al.(1963)]

Diameter of mean sphere size d (cm)	Volume concentration c	Experimental data of Permeability k (cm^2)	Permeability obtained here k (cm^2)
0.2	0.60	34.6×10^{-6}	43.3×10^{-6}
0.2	0.62	24.5×10^{-6}	37.4×10^{-6}
0.2	0.63	22.0×10^{-6}	34.7×10^{-6}
0.14	0.62	15.7×10^{-6}	18.3×10^{-6}
0.092	0.63	5.70×10^{-6}	7.35×10^{-6}
0.039	0.64	1.03×10^{-6}	1.23×10^{-6}

3.3 Parametric study of the trial functions

In the previous section we determined the values A_i of $\exp\{-A_i(R-1)\}$ in the trial function by considering the value of B of $\exp(-BR)$ in the Brinkman solution (3.12). For instance, by the Brinkman solution, $B \simeq 1.94$ for $c = 0.2$ and $B \simeq 5.19$ for $c = 0.4$, so that we selected $A_1 = 2, A_2 = 6$ in the previous section. However there must be more appropriate A_i and z for the trial functions. Therefore we conducted several Galerkin calculations for

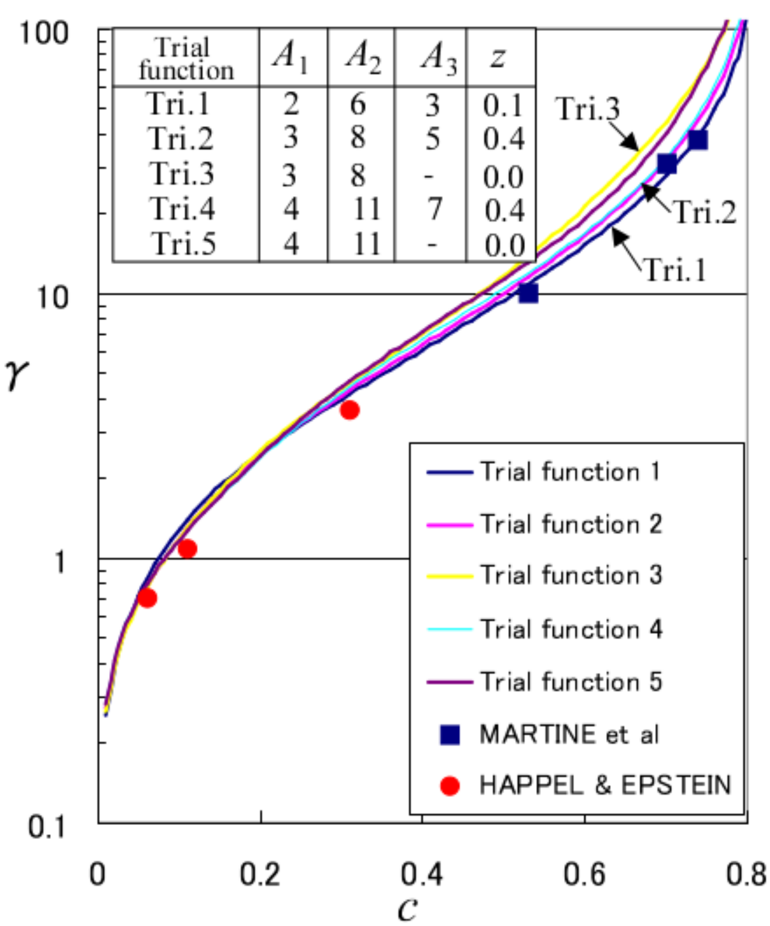

Figure 3.2: Comparison of γ for various trial functions.

the following trial functions:

$$\Phi_t = 1 - \frac{5}{2R^3} + \frac{3}{2R^5} + a_1 R^2 (R-1)^2 \mathrm{e}^{-A_1(R-1)}$$
$$+ a_2 R^2 (R-1)^2 \left[\mathrm{e}^{-A_2(R-1)} + z\mathrm{e}^{-A_3(R-1)}\right]. \qquad (3.41)$$

The results of the γ by calculations with several trial functions are shown in Fig.3.2, and A_i's and z are listed on the upper left corner of the figure. Trial fnction 1 is an eqn.(3.25). The accuracy of those traial functions are

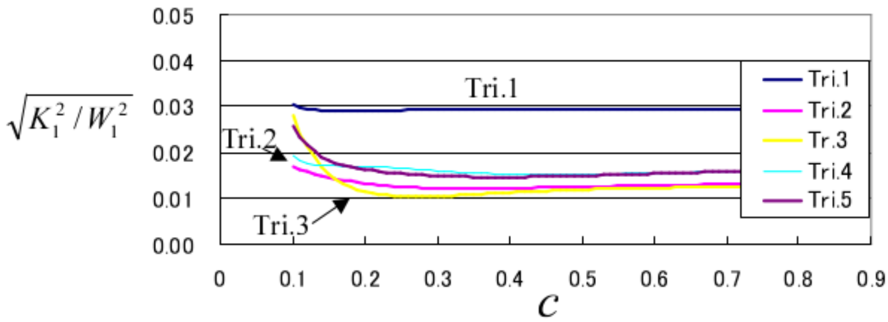

Figure 3.3: Values of $\sqrt{K_1^2/W_1^2}$ for various trial functions.

estimated by the values of $\sqrt{K_1^2/W_1^2}$ defined by in the conclusion. From Fig.3.3 trial function 2 or 3 should be the most appropriate solution of fundamental equation (3.21) from the point of the accuracy.

Chapter 4

Permeability of a Random Array of Parallel Infinite Cylindrical Rods

In this chapter we discuss the permeability of a fibrous material. Here, the material is supposed to be composed of a random array of cylindrical rods. For this case two types of flow are considered; one is a longitudinal flow parallel to an axis of the cylindrical rod and the other is a transverse one perpendicular to them. (See Fig.1.3)

We consider the slow flow past a random array of N cylindrical rods, all of radius a. And it is assumed that the rods are distributed in a statistically homogeneous manner. To restrict the problem to the two-dimensional one we also assume that these rods are rigidly fixed in parallel and are infinite in length. Position vectors of centers of the rods lie in a plane; the rods are circular cylinders perpendicular to the plane, and distributed with mean number of density λ per unit area of it. Then, the volume fraction of the rods, c, is $\pi\lambda a^2$.

The fluid flowing through the array of the rods is governed by the equation of continuity and Stokes' equation. Inside the rod we define $\mathbf{u} = \mathbf{0}$, $\nabla p = 0$, where $\mathbf{u}$ is the velocity and p the pressure of the fluid. By a similar procedure as in section 2 we obtain the same averaged equations as those for those for the random array of spheres:

i) For an unconditional averaged flow

$$\left\{ \begin{array}{c} \nabla. < \mathbf{U}(\mathbf{r}) >= 0, \\ -\nabla < Hp(\mathbf{r}) > +\mu\nabla^2 < \mathbf{U}(\mathbf{r}) > -\mu\alpha^2 < \mathbf{U}(\mathbf{r}) >= \mathbf{0}, \end{array} \right. \tag{4.1}$$

ii) For a conditional averaged flow

$$\left\{ \begin{array}{c} \nabla. < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >= 0, \\ -\nabla < Hp(\mathbf{r}|\mathbf{r}_1) > +\mu\nabla^2 < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) > -\mu\alpha^2 Q_{\mathrm{C}} < \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >= \mathbf{0}. \end{array} \right. \tag{4.2}$$

In the above equations we use the same notations as in section 2, and

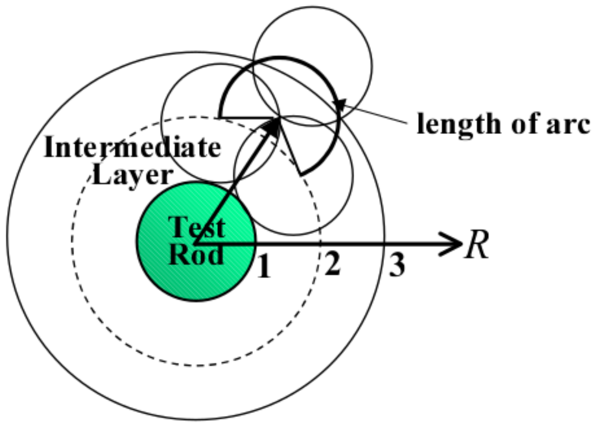

Figure 4.1: Intermediate layer around a test rod.

the function Q_{C} represents the dependence on the radial distance of the intermediate layer for the cylindrical rods. In this case Q_{C} is the value of the ratio of the length of the arc whose radius is a and centered at $\mathbf{r}$, to $2\pi a$. (See Fig.4.1) Then, from the geometry of the location of the rods around test rods, we obtain

$$Q_{\mathrm{C}}(R) = \begin{cases} \dfrac{1}{\pi} \arccos \dfrac{3 - R^2}{2R} & \cdots \quad (1 \le R < 3), \\ 1 & \cdots \quad (R \ge 3), \end{cases} \tag{4.3}$$

where $R = |\mathbf{r} - \mathbf{r}_1|/a$.
The self-consistency condition for this case is as follows: Since λ stands for the mean number density of the rods, $\mathbf{f}$ the drag of the rod, and $\mathcal{F}(\mathbf{r})$ corresponds to a mean resistance per unit volume, we require

$$\lambda \mathbf{f} = \mathcal{F} = \mu \alpha^2 < \mathbf{U} > . \tag{4.4}$$

4.1 Permeability for the longitudinal flow

We shall discuss the permeability for the longitudinal flow (along the axis of the rod) here. We choose a cylindrical coordinates (r, θ, z), so that the z-axis coincides with axis of the test cylindrical rod. We introduce the dimensionless radial distance, R, defined by $R = r/a$. The boundary conditions for eqs.(4.2) are

$$< \mathbf{U}(\mathbf{r}|\mathbf{O}) >= \mathbf{0}, \qquad \text{on the test rod}, \tag{4.5}$$

$$< \mathbf{U}(\mathbf{r}|\mathbf{O}) >= \mathbf{U}_0 = U_0\mathbf{e}_z, \qquad \text{at} \quad \text{infinity}, \tag{4.6}$$

where $\mathbf{e}_z$ is the unit vector in the z-direction and $\mathbf{U}_0$ is a constant. In this case, the situation is rather simple because the velocity field is effectively one-dimensional. Taking into account of the condition at infinity we choose

$$< \mathbf{U}(\mathbf{r}|\mathbf{r}_1) >=< \mathbf{U}(\mathbf{R}|\mathbf{O}) >= U_0\psi(R)\mathbf{e}_z, \tag{4.7}$$

which satisfies the equation of continuity. Then by substituting eq.(4.7) for an equation of motion in (4.2) we obtain $\partial < Hp > /\partial R = 0, \partial < Hp > /\partial\theta = 0$ and

$$\frac{\mathrm{d}^2\psi(R)}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}\psi(R)}{\mathrm{d}R} - \alpha^2 a^2 Q_{\mathrm{C}}\psi(R) = \frac{a^2}{\mu U_0}\frac{\partial}{\partial z} < Hp > \tag{4.8}$$

From above equations we can find $\partial < Hp > /\partial z$ =constant, so that the pressure is expressed by

$$< Hp(\mathbf{R}|\mathbf{O}) >= -\mu U_0\frac{B^2}{a^2}z, \tag{4.9}$$

where $B = \alpha a$ and we neglected the arbitrary constant because it does not have any contribution to the drag. Putting eq.(4.9) in eq.(4.8), one finds

$$\ddot{\psi} + \frac{1}{R}\dot{\psi} - B^2 Q_{\mathrm{C}}\psi + B^2 = 0. \tag{4.10}$$

The boundary conditions are rewritten by

$$\psi(1) = 0, \tag{4.11}$$

$$\psi \longrightarrow 1 \quad \text{as} \quad R \to \infty. \tag{4.12}$$

The average stress tensor can also be written in terms of ψ and the pressure given by eq.(4.9):

$$< H\mathbf{T}(\mathbf{R}|\mathbf{O}) >= \mu U_0\frac{B^2}{a^2}z\mathbf{I} + \mu\frac{U_0}{a}\dot{\psi}(\mathbf{e}_R\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_R). \tag{4.13}$$

By integrating the stress on the surface of a cylindrical rod, we obtain the mean drag, $\mathbf{f}$, per unit length of the rod:

$$\mathbf{f} = \int_S < H\mathbf{T} > \cdot\mathbf{n}_1\mathrm{d}S = 2\pi\mu\dot{\psi}\mathbf{U}_0, \tag{4.14}$$

where $\mathbf{n}_1$ is the unit outward normal vector, and S is the surface of it with unit length. Since a mean resistance per unit volume is $\mu\alpha^2\mathbf{U}_0$, the self-consitency condition becomes

$$\lambda\mathbf{f} = \mu\alpha^2\mathbf{U}_0,$$

and we obtain

$$B^2 = 2c\dot{\psi}. \tag{4.15}$$

Therefore if ψ were obtained, the permeability k (or B) would be calculated from eqs.(2.38) and (4.15). As is employed in section 3, we shall adopt the direct method to get an approximate solution of eq.(4.10). A trial function ψ_t is assumed to be

$$\psi_t = 1 - \frac{1}{R^2} + \bar{a}_1(R-1)\mathrm{e}^{-(R-1)} + \bar{a}_2(R-1)\mathrm{e}^{-2(R-1)}, \tag{4.16}$$

which satisfies the boundary conditions (4.11) and (4.12). By the Galerkin method the coefficients a_j's are determined:

$$\int_1^{R_F} L_2(\psi)\bar{\omega}_j \mathrm{d}R = 0, \tag{4.17}$$

where $L_2(\psi)$ denotes eq.(4.10) and $\bar{\omega}_j$ represents $\bar{\omega}_1 = (R-1)\exp[-(R-1)]$ and $\bar{\omega}_2 = (R-1)\exp[-2(R-1)]$. It should be noted that

$$\bar{a}_j\bar{\omega}_j(1) = 0,$$

$$\bar{a}_j\bar{\omega}_j \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty.$$

We set $R_F = 100$, and after the numerical calculation we find

$$\begin{cases} \bar{a}_1 = [-0.0027783B^4 + 0.072309B^2 - 0.025415]/\bar{\Lambda}, \\ \bar{a}_2 = [0.056187B^4 - 0.10352B^2 - 0.0038565]/\bar{\Lambda}, \\ \text{where} \\ \bar{\Lambda} = 0.0054133B^4 + 0.052022B^2 + 0.024289. \end{cases} \tag{4.18}$$

Since eq.(4.15) can be rewritten in terms of ψ_t:

$$B^2 = 2c\dot{\psi}_t(1) = 2c(2 + \bar{a}_1 + \bar{a}_2), \tag{4.19}$$

then by substitution of eq.(4.18) into it, and after a cancellation of the denominator of $\bar{a}_j$ and by rearrangement, we obtain an equation of degree six for B

$$B^6 + (9.6100 - 23.732c)B^4 + (4.4869 - 26.909c)B^2 - 7.1330c = 0. \tag{4.20}$$

Root of the algebraic equation for each value of c is shown in Table 4.1.
Figure 4.2 shows a comparison of γ obtained here with that of spheres calculated in Chap.3.

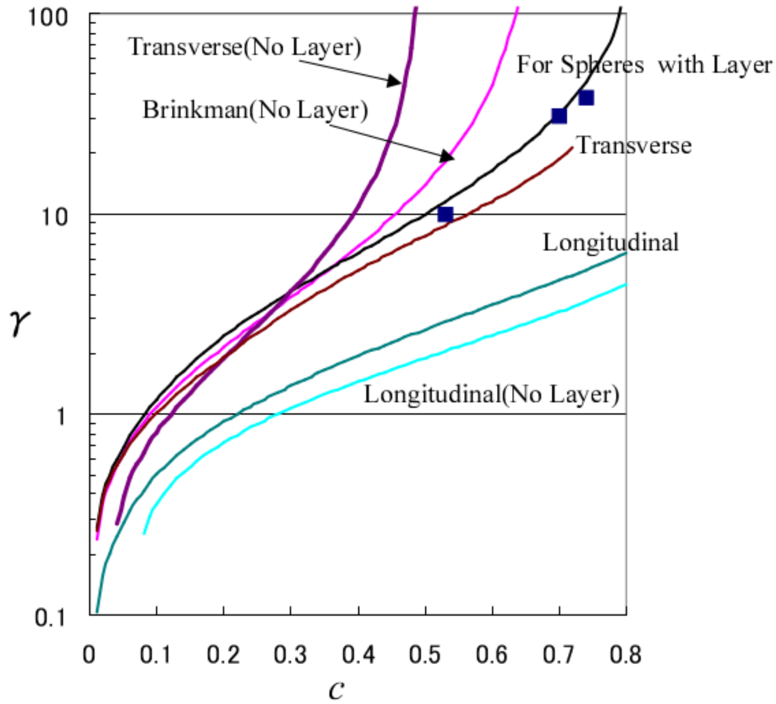

Figure 4.2: Comparison of γ for cylindrical rods with that for Spheres [28]. Here, "No Layer" means no interemediate layer. And see appendix C.

4.2 Permeability for the transverse flow

Here we shall consider the permeability for the transverse flow. Fundamental equations and the boundary conditions are eqs.(4.1)-(4.6) We choose the same coordinate frame as in the previous section. For the transverse flow the velocity, $\mathbf{U}_0$, at infinity can be written as follows:

$$\mathbf{U}_0 = U_0(\cos\theta \mathbf{e}_R - \sin\theta \mathbf{e}_\theta), \tag{4.21}$$

where $\mathbf{e}_R$ and $\mathbf{e}_\theta$ are the unit vectors in the R and θ directions, respectively. In the light of eq.(4.21), we assume a solution of the fundamental equations as follows:

$$< \mathbf{U}(\mathbf{R}|\mathbf{O}) >= U_0[\Psi(R)\cos\theta \mathbf{e}_R - \{\Psi(R) + R\dot{\Psi}(R)\}\sin\theta \mathbf{e}_\theta], \tag{4.22}$$

and

$$< Hp(\mathbf{R}|\mathbf{O}) >= -\mu U_0 \frac{B^2}{a}\Omega\cos\theta. \tag{4.23}$$

Table 4.1: Values of B and γ

Volume concentration c	B	γ
0.0001	0.01261	0.01261
0.001	0.03992	0.03994
0.01	0.12780	0.1284
0.05	0.3018	0.3090
0.1	0.4600	0.4848
0.2	0.7742	0.8656
0.3	1.161	1.388
0.4	1.629	2.103
0.5	2.110	2.984
0.6	2.553	4.037
0.7	2.952	5.390
0.8	3.313	7.468
0.9	3.643	11.52

The velocity in eq.(4.22) satisfies the equation of continuity. Then, putting eqs.(4.22) and (4.23) in the equation of motion, we find a set of basic equations governing $\Psi(R)$ and $\Omega(R)$

$$\ddot{\Psi} + \frac{3}{R}\dot{\Psi} + B^2\dot{\Omega} - B^2 Q_{\mathrm{C}}\Psi = 0, \tag{4.24}$$

and

$$\ddot{\Omega} + \frac{1}{R}\dot{\Omega} - \frac{1}{R^2}\Omega - B^2\dot{Q}_{\mathrm{C}}\Psi = 0, \tag{4.25}$$

with the boundary conditions

$$\Psi(1) = \dot{\Psi}(1) = 0, \tag{4.26}$$

$$\Psi \longrightarrow 1 \quad \text{as} \quad R \to \infty. \tag{4.27}$$

The average stress tensor can be written in terms of Ψ and Ω

$$< H\mathbf{T}(\mathbf{R}|\mathbf{O}) >= \mu U_0 \frac{B^2}{a}\Omega\cos\theta\mathbf{I}$$

$$+\mu\frac{U_0}{a}[2\dot{\Psi}\cos\theta\mathbf{e}_R\mathbf{e}_R - (\dot{\Psi} + R\ddot{\Psi})\sin\theta(\mathbf{e}_\theta\mathbf{e}_R + \mathbf{e}_R\mathbf{e}_\theta) - \dot{\Psi}\mathbf{e}_\theta\mathbf{e}_\theta]. \tag{4.28}$$

The drag per unit length is given by

$$\mathbf{f} = \mathbf{f}_{\mathrm{Pressure}} + \mathbf{f}_{\mathrm{Friction}} = \mu\mathbf{U}_0[B^2\Omega(1) + \ddot{\Psi}(1)]. \tag{4.29}$$

Then the self-consistency condition is written by

$$c[B^2\Omega + \ddot{\Psi}] = B^2, \tag{4.30}$$

from which we can obtain the values of B, provided that the basic equations (4.24) and (4.25) are solved. However, it is impossible to get the exact solutions of the set of coupled equations. Furthermore two difficulties arise in eq.(4.25); first the function $\dot{Q}_{\rm C}$ diverges at $R = 1$ and $R = 3$, that is

$$\dot{Q}_{\rm C}(R) = \frac{R^2+3}{\pi\sqrt{(3-R)(3+R)(R-1)(R+1)}}, \tag{4.31}$$

second we can not obtain the solution by direct integration due to the inhomogeneous term $\dot{Q}_{\rm C}\Psi$. The behavior of $\dot{Q}_{\rm C}\Psi$ near the point $R = 1$ and 3 are as follows: Since $\dot{Q}_{\rm C} \sim (\Delta R)^{-1/2}$ and $\Psi \sim (\Delta R)^2$, due to the boundary condition[1], the source term $\dot{Q}_{\rm C}\Psi \to 0$ as $R \to 1$, where $\Delta R = R - 1$. On the other hand, if we define $\Delta R' = R - 3$, then $\dot{Q}_{\rm C} \sim (\Delta R')^{-1/2}$ and $\Psi \sim 1$, so that $\dot{Q}_{\rm C}\Psi \to \infty$ as $R \to 3$. Hence we find the inhomogeneous term is singular only at $R = 3$. But the point is physically ordinary point in the conditional averaged flow field. To carry out our scheme we modify $\dot{Q}_{\rm C}$ in the following form

$$\dot{Q}_{\rm M}(R) = 2\left\{{\rm e}^{-8(R-1)} + {\rm e}^{10(R-3)} + \frac{3}{10}\right\}, \tag{4.32}$$

which is nearly equal to $\dot{Q}_{\rm C}$ for $1 < R < 3$, but is regular at $R = 1$ and 3. Thus, we get a regular inhomogeneous term $\dot{Q}_{\rm M}$ for $1 \leq R \leq 3$. Comparison of $\dot{Q}_{\rm C}$ with $\dot{Q}_{\rm M}$ is shown in Fig.4.3.

Here we assume the trial function $\Psi_{\rm t}$ such that

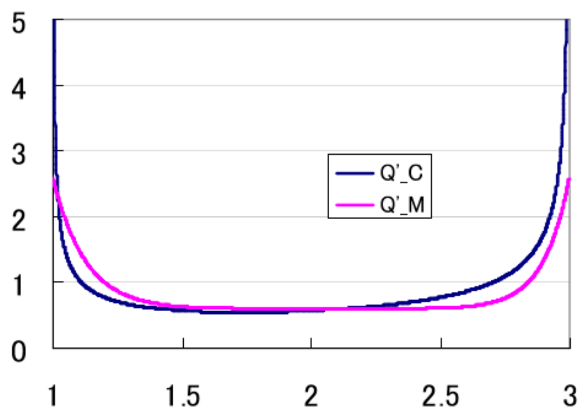

Figure 4.3: Comparison of $\dot{Q}_{\rm C}$ with $\dot{Q}_{\rm M}$.

[1]The Ψ will be expanded at $R = 1$ as $\Psi(R) \approx a_0 + a_1(R-1) + a_2(R-1)^2 + \cdots +$. However, due to the boundary condition $\Psi(1) = \dot{\Psi}(1) = 0$, $\Psi \approx a_2(R-1)^2 + o(\Delta R^2)$.

$$\Psi_{\mathrm{t}}=1-\frac{5}{3}\mathrm{e}^{-2(R-1)}+\frac{2}{3}\mathrm{e}^{-5(R-1)}+(R-1)^2\left\{\hat{a}_1\mathrm{e}^{-3(R-1)}+\hat{a}_2\mathrm{e}^{-7(R-1)}\right\} \quad (4.33)$$

which satisfies the boundary condition (4.26) and (4.27). By substitution of $\dot{Q}_{\mathrm{M}}$ and Ψ_{t} into eq.(4.26), we get the solution of eq.(4.25) for $1 \le R < 3$

$$\Omega_{-}=\hat{\Xi}R+\frac{\hat{S}}{R}+R\int_1^R\frac{\dot{Q}_{\mathrm{M}}\Psi_{\mathrm{t}}}{2}\mathrm{d}R'-\frac{1}{R}\int_1^R\frac{R'^2\dot{Q}_{\mathrm{M}}\Psi_{\mathrm{t}}}{2}\mathrm{d}R' \quad (4.34)$$

where the integrations are easy but tedious. For $R \ge 3$

$$\Omega_{+}=R+\frac{\hat{\Sigma}}{R}. \quad (4.35)$$

In the above equations $\hat{\Xi}$, $\hat{S}$ and $\hat{\Sigma}$ are unknown constants. First, we can determine $\hat{S}$ by the same way as in the section 3.2: Since the drag $\mathbf{f}$ given by eq.(4.29) is written in

$$\mathbf{f}=\mu\mathbf{U}_0\pi 2\hat{S},$$

we find from the self-consistency condition

$$\hat{S}=\frac{1}{2c}. \quad (4.36)$$

On the other hand, $\hat{\Xi}$ and $\hat{\Sigma}$ can be obtained by continuity condition at $R=3$. From a continuity of $R-R$ component of the stress $< H\mathbf{T}(\mathbf{R}|\mathbf{O}) >$

$$\Omega_{-}(3)=\Omega_{+}(3), \quad (4.37)$$

where we used $\Psi_{\mathrm{t}}(R)$ proposed by eq.(4.33). All other components of the stress should be continuous at the point, so we find the condition by considering eq.(4.24):

$$\dot{\Omega}_{-}(3)=\dot{\Omega}_{+}(3). \quad (4.38)$$

Thus we can determine $\hat{\Xi}$ and $\hat{\Sigma}$ by eqs.(4.37) and (4.38):

$$\begin{aligned}\hat{\Xi}&=0.69655-0.023576\hat{a}_1-0.0023425\hat{a}_2,\\ \hat{\Sigma}&=\frac{1}{2c}-1.9357-0.094545\hat{a}_1-0.0045444\hat{a}_2.\end{aligned} \quad (4.39)$$

Then the Galerkin method is used to get an approximate solution of eq.(4.24), that is,

$$\int_1^{R_{\mathrm{F}}}L_3(\Psi)\hat{\omega}_j\mathrm{d}R=0, \quad (4.40)$$

where $\hat{\omega}_1=(R-1)^2\exp[-3(R-1)]$ and $\hat{\omega}_2=(R-1)^2\exp[-7(R-1)]$. (cf., it should be noted that $\hat{\omega}_j(1)=0$ and $\hat{\omega}_j\to 0$ as $R\to\infty$.) Setting $R_{\mathrm{F}}=100$, we find after computation

$$\begin{cases} \hat{a}_1 = \dfrac{1}{\hat{\Lambda}}\left[\left(3.2562 - \dfrac{1.0718}{c}\right)B^4 + \left(149.73 - \dfrac{59.567}{c}\right)B^2 - 0.81766\right], \\ \hat{a}_2 = \dfrac{1}{\hat{\Lambda}}\left[\left(3.8879 - \dfrac{7.1466}{c}\right)B^4 + \left(457.06 - \dfrac{131.78}{c}\right)B^2 - 1009.1\right], \\ \hat{\Lambda} = (2.6597B^4 + 3.7524B^2 + 0.47142) \times 10^7. \end{cases} \tag{4.41}$$

We have had the approximate solutions $\Psi_t(R)$ and $\Omega_-(R)$, so the drag $\mathbf{f}$ is obtained. The self-consistency condition for determination of B becomes

$$\left(c\hat{\Xi} - \frac{1}{2}\right)B^2 + c\left[10 + 2(\hat{a}_1 + \hat{a}_2)\right] = 0, \tag{4.42}$$

where $\hat{\Xi}$ is given by eq.(4.39), and $\hat{a}_1$ and $\hat{a}_2$ have been obtained in (4.41). Therefore eq.(4.42) turns out to be the equation of degree six for B after a cancellation of the denominator of $\hat{a}_j$:

$$\begin{cases} B^6 + \dfrac{216.55 - 235.42c}{W}B^4 + \dfrac{3908.6 - 13527c}{W}B^2 - \dfrac{25676c}{W} = 0, \\ \text{where} \\ W = 1.0061 - 1.4017c. \end{cases} \tag{4.43}$$

Root of the above equation and values of γ are shown for each value of c in Table.4.2, and γ is shown in Fig.4.2.

Table 4.2: Values of B and γ

Volume concentration c	B	γ
0.0001	0.002563	0.002564
0.001	0.008118	0.008118
0.01	0.2604	0.2617
0.1	0.9684	1.021
0.2	1.730	1.934
0.3	2.770	3.310
0.4	4.051	5.230
0.5	5.507	7.788
0.6	7.311	11.56
0.7	10.27	18.74

4.3 Parametric study of the trial functions for a transverese flow

As well as the section 3.3 we also conducted several Galerkin calculations for the following trial functions in order to find a more accurate approximate solution:

$$\Psi_t = 1 - \frac{5}{3}\mathrm{e}^{-2(R-1)} + \frac{2}{3}\mathrm{e}^{-5(R-1)} + (R-1)^2\left\{a_1\mathrm{e}^{-A_1(R-1)} + a_2\mathrm{e}^{-A_2(R-1)}\right\}. \tag{4.44}$$

The results of the γ by calculations with several trial functions are shown in

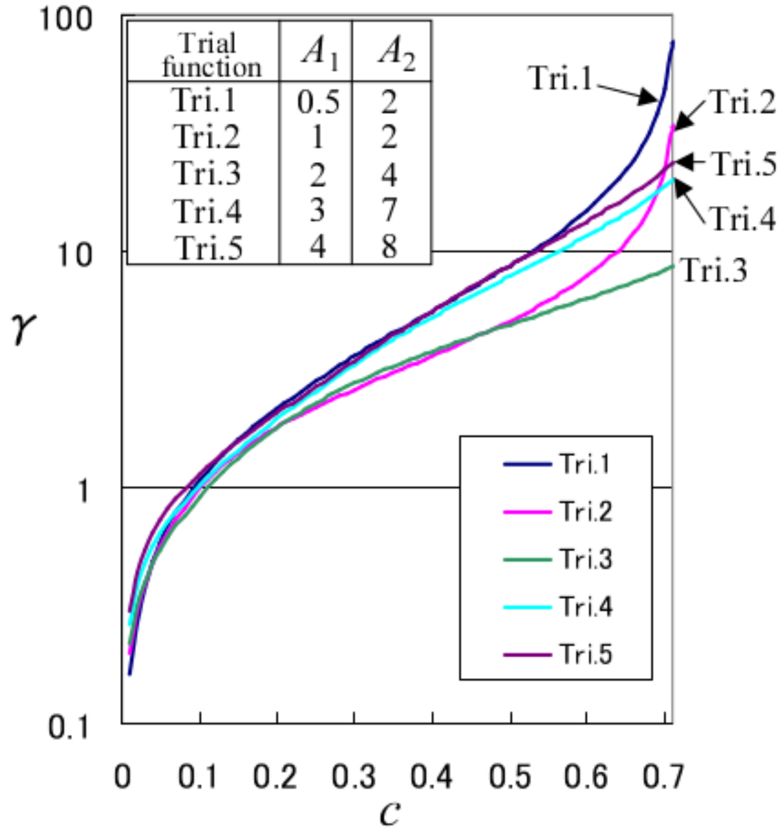

Figure 4.4: Comparison of γ for various trial functions.

Fig.4.4, and A_i's are listed on the upper left corner of the figure. Trial fnction 4 is an eqn.(4.33) and is the most accurate one of those with considering the values of $\sqrt{K_3^2/W_3^2}$ defined by in the conclusion.

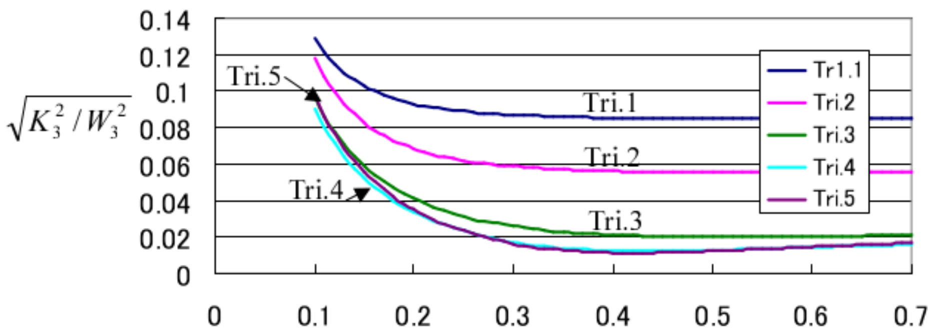

Figure 4.5: Values of $\sqrt{K_3^2/W_3^2}$ for various trial functions for transverse flow.

Chapter 5

Conclusion

As is mentioned in chapter 1, the method by the ensemble average has not been able to predict a valid permeability for the densely packed beds of spheres. Therefore it seems as if the approach by the method could not be applied to the case. However, by introducing the idea of the intermediate layer with using the ensemble average , proposed by Buevich & Marcov, we obtained a reasonable permeability : the permeability obtained here does not become zero at $c = 2/3$. In other words, the critical concentration at which the fluid can not permeate shifts to a larger one than that calculated by neglecting the layer. This is natural because the introduction of the layer means the decrease of the Darcy resistance around the test body in the conditional averaged field. (So fluid flows more easily through the porous mass.) Actually our result agrees with the experimental data for the densely packed beds.

On the other hand, in region $c < 0.4$ the non-dimensional quantity γ is larger than γ_{Br} which is calculated by neglecting the layer. We may say that the result is due to the following fact. By taking account of the layer the Darcy resistance decreases near the test sphere, so that the fluid passes more easily around it. Then the velocity around the test sphere becomes large, while the velocity on the sphere must be zero due to no-slip condition. Therefore the change in the velocity of a mean flow near it becomes larger than that when neglecting the intermediate layer.This causes a larger frictional drag on the test sphere. The same effect is seen in the case of the void-region of the resistance around the test spher (cf., Appendix B).

In chapter 4 we discussed the permeability of the array of cylindrical rods, which is the model of the fibrous material, and we found that the permeability for the transverse flow is smaller than that of the longitudinal flow entire region of c. When considering an intermediate layer the permeability for the transverse flow is smaller than that for neglecting the layer. In the longitudinal flow, however, the permeability considering the layer is smaller than that for neglecting in entire region of c. That is, when considering the intermediate layer it becomes difficult to go through the fiborus material for the longitudinal flow. And as is seen in Fig.4.3 the non-dimensional quantity γ diverges at $c = 1/2$ in the transverse flow (cf., Appendix C), and at $c = 2/3$ for the spheres when neglecting the intermediate layer.

The drag on the solid material is due to the pressure distribution and

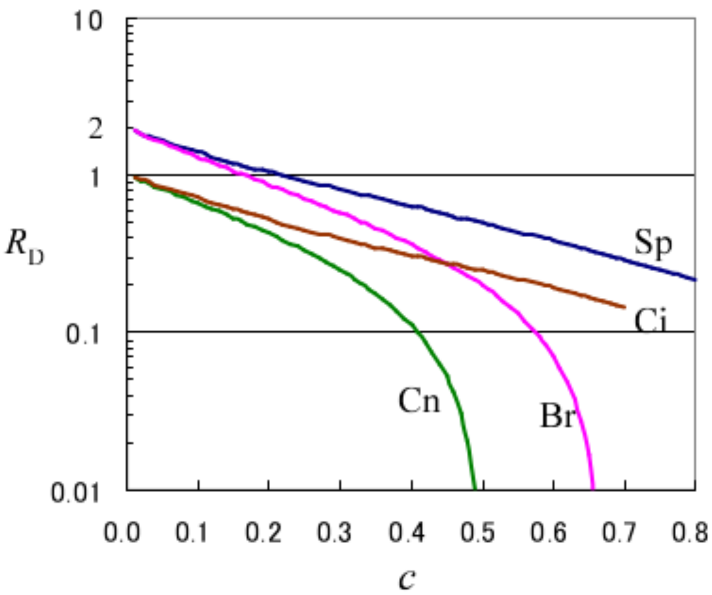

Figure 5.1: The ratio , $R_{\rm D}$, of the viscous drag to the pressure drag. **Sp**:the ratio for the packed bed of spheres with intermediate layer, **Br**: that for Brinkman, **Ci**: that for transverse flow around cylinder with intermediate layer, **Cn**: that for transverse flow neglecting the layer.

an existence of shear. In Fig.5.1 we show a ratio, $R_{\rm D}$, of the viscous drag to the pressure drag: $R_{\rm D}$=(Viscous drag)/(Drag by pressure). For instance, $\mathbf{U}_0$ components of a drag on the test sphere are $\mathbf{f}_P$ by the pressure,

$$\mathbf{f}_P = \int_S \mu U_0 \frac{B^2}{a} \Pi(1) \cos^2\theta \mathrm{d}S,$$

and $\mathbf{f}_\mu$ by the shearing stress:

$$\mathbf{f}_\mu = \int_S \mu \frac{U_0}{2a} \ddot{\Phi}(1) \sin^2\theta \mathrm{d}S.$$

Therefore, the drag on the sphere is

$$\mathbf{f} = \mathbf{f}_P + \mathbf{f}_\mu = \mu U_0 B^2 \Pi(1) \frac{4a\pi}{3} + \mu U_0 \ddot{\Phi}(1) \frac{4a\pi}{3}$$

$$= \mu U_0 B^2 \frac{4a\pi}{3}[\Pi(1) - \dot{\Pi}(1)]$$

where the boundary condition (3.23) is used, while the ratio, $R_{\rm D}$, considering the intermediate layer

$$R_{\rm D} = \frac{-\dot{\Pi}(1)}{\Pi(1)} = \frac{2/(3c) - \Xi}{1/(3c) + \Xi},$$

where Ξ is obtained by (3.33). Especially fot the Brinkman case, $R_{\rm D}$, is as follows:

$$R_{\rm D} = \frac{-\dot{\chi}(1)}{\chi(1)} = \frac{2(3+3B)}{3+3B+3B^2},$$

where the solutions (3.12) and (3.13) are used, and B is defined by (3.16). It is well known that $R_{\rm D} = 2$ for the Stokes drag on a sphere, and we can also say that $c \to 0; R_{\rm D} \to 2$ for the Brinkman model and the swarm of the spheres. On the other hand, we can find from Fig.5.1 the drag is (mainly) due to the pressure distribution in the case of densely packed beds of spheres. In the case of the transverse flow for the cylindrical rods, when considering the layer:

$$R_{\rm D} = \frac{-\dot{\Omega}(1)}{\Omega(1)} = \frac{1/(2c) - \hat{\Xi}}{1/(2c) + \hat{\Xi}},$$

where $\hat{\Xi}$ is defined by eqn.(4.39). And when the intermediate layer is neglected,

$$R_{\rm D} = \frac{-\dot{\Omega}(1)}{\Omega(1)} = \frac{2K_1}{2B_T K_0 + 2K_1} = \frac{1-2c}{1+2c}.$$

Here we used eqs.(C.11) and eq.(C.12).

In this article we use the Galerkin method to get an approximate solutions of the fundamental equations. Then a question will arise about the accuracy of the Galerkin method, so that we solve Brinkman equation approximately by using the method and calculate the values[1] of B from the solution. Then we shall compare the exact $B_{\rm E}$ obtained in section 3.1 with the approximate $B_{\rm A}$. The brief procedure and a good agreement between those can be seen in appendix D. The purpose of this calculation, however, is to assure the necessity for the solution.

Whether the approximate solutions (or the trial functions) converge or not is a matter for argument. We can say nothing about this question because we can not choose the trial functions from a system of orthogonal functions. Therefore, we must make a careful choice in employing the trial functions. It should be noted how we choose the solutions in chapter 3 and chapter 4: Among some appropriate trial functions we selected the most allowable

[1] We denote it as $B_{\rm A}$

functions which minimize the quantities, $\sqrt{K_j^2/W_j^2}$, defined by as follows.

In chapter 3,

$$K_1^2 = \int_1^{R_\mathrm{F}} \left[\ddot{\Phi}_\mathrm{t} + \frac{4}{R}\dot{\Phi} + B^2\dot{\Pi}_\pm - B^2 Q_\mathrm{S}\Phi_\mathrm{t}\right]^2 \mathrm{d}R = \int_1^{R_\mathrm{F}} \Big\{ L_1[\Phi_\mathrm{t}]\Big\}^2 \mathrm{d}R,$$

$$W_1^2 = \int_1^{R_\mathrm{F}} \left[\ddot{\Phi}_\mathrm{t}^2 + \frac{16}{R^2}\dot{\Phi}^2 + B^4\dot{\Pi}_\pm^2 + B^4 Q_\mathrm{S}^2\Phi_\mathrm{t}^2\right] \mathrm{d}R.$$

In chapter 4, for a longitudinal flow:

$$K_2^2 = \int_1^{R_\mathrm{F}} \Big\{ L_2[\psi_\mathrm{t}]\Big\}^2 \mathrm{d}R,$$

$$W_2^2 = \int_1^{R_\mathrm{F}} \left[\ddot{\psi}_\mathrm{t}^2 + \frac{1}{R^2}\dot{\psi}^2 + B^4 + B^4 Q_\mathrm{C}^2\psi_\mathrm{t}^2\right] \mathrm{d}R,$$

for a transverse flow:

$$K_3^2 = \int_1^{R_\mathrm{F}} \Big\{ L_3[\Psi_\mathrm{t}]\Big\}^2 \mathrm{d}R,$$

$$W_3^2 = \int_1^{R_\mathrm{F}} \left[\ddot{\Psi}_\mathrm{t}^2 + \frac{1}{R^2}\dot{\Psi}^2 + B^4\dot{\Omega}_\pm^2 + B^4 Q_\mathrm{C}^2\Psi_\mathrm{t}^2\right] \mathrm{d}R.$$

Those values are shown in Table 5.1.

Table 5.1: Values of $\sqrt{K_j^2/W_j^2}$

Volume concentration c	$\sqrt{K_1^2/W_1^2}$	$\sqrt{K_2^2/W_2^2}$	$\sqrt{K_3^2/W_3^2}$
0.1	0.028	0.120	0.090
0.2	0.040	0.038	0.033
0.3	0.022	0.033	0.017
0.4	0.015	0.034	0.012
0.5	0.013	0.030	0.013
0.6	0.012	0.025	0.014
0.7	0.012	0.020	0.016
0.8	0.013	0.015	—
0.9	0.013	0.012	—

It should be noted that these criterions are not the sufficient condition but the necessary condition.

Acknowledgement

The author would like to thank Professor Ken-ichi Kusukawa, former President of Tokyo Metropolitan University, for many valuable comments on this thesis and continual interest.

He also wishes to thank Professor Tsuguo Takahashi for his guidance to the fluid-particle system and for stimulating discussion.

Appendix A

Darcy's law

Pioneering work on the flow through the porous medium was made by Darcy in 1856. He conducted an experiment on vertical pipe of cross-sectional area A filled with sand, under conditions simulated by Fig.A.1. From his investigations of the flow through horizontal stratified beds of sand, Darcy concluded that the flow rate Q was proportional to $h_2 - h_1$, inversely proportional to the length of the flow path, ΔL, and proportional to a coefficient K, depending on the nature of the sand. (He failed to recognize that K depends on properties of the fluid as well as on that of medium.)

Darcy's law may be expressed as

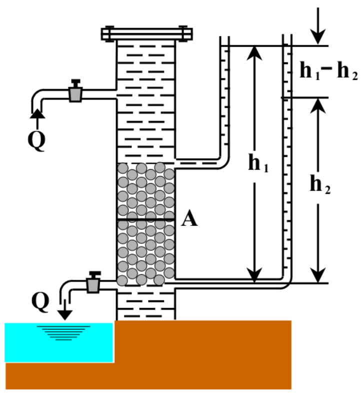

Figure A.1: Apparatus to demonstrate Darcy's law.

$$Q = KA\frac{h_1 - h_2}{\Delta L}, \tag{A.1}$$

where k is the permeability and h equals the piezometric head. Now let $K = k\rho g/\mu$, then eq.(A.1) becomes

$$\frac{Q}{A} = -\frac{k}{\mu}\frac{(\rho g h_2 - \rho g h_1)}{\Delta L} = -\frac{k}{\mu}\frac{\Delta P}{\Delta L}, \tag{A.2}$$

where ρ denotes the density of the fluid and g the gravitational acceleration and ΔP is the pressure drop. The above equation in vectorial form may be written by using flux, $\mathbf{q}$, per unit area:

$$\mathbf{q} = -\frac{k}{\mu}\nabla P. \tag{A.3}$$

Finally we show a recent experimental apparatus [23] to measure the permeability in Fig.A.2. Particular care is taken with respect shape and size of the spherical particles (precision ground stainless steel balls of 1[mm] diameter). The spheres are randomly packed into a heavy-wall brass cylinder of 50.8[mm] inner diameter, submerged under distilled water to preclude the inclusion of air bubbles into the system. A steady stream of distilled water is passed through the system; its flow rate is set and controlled by means of a hydraulic flow controller. A pressure transducer, periodically calibrated against a high precision differential manometer, is used to measure the pressure difference.

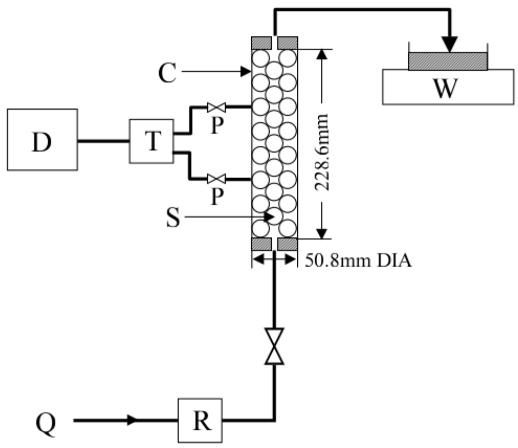

Figure A.2: Equipment used for the experimental determination of a pack of spheres. C: cylindrical brass cell, S: stainless steel spheres, Q: distilled water supply, W: weighing scale, P: pressure taps (spaced 76.2 mm apart), T: pressure transducer, D: digital voltmeter, R: constant flow regulator.

Appendix B

A void-region model around the test sphere

As is mentioned in chapter 1 a void region of the sphere-center is formed around the test sphere because of the character of the conditional probability density function $P(\mathbf{r}|\mathbf{r}_1)$:

$$P(\mathbf{r}|\mathbf{r}_1) = \begin{cases} 0 \quad \text{for} \quad a \leq |\mathbf{r} - \mathbf{r}_1| < 2a, \\ \\ 1 \quad \text{for} \qquad |\mathbf{r} - \mathbf{r}_1| \geq 2a. \end{cases} \tag{B.1}$$

Putting $P(\mathbf{r}_2|\mathbf{r}_1) = P(\mathbf{r}|\mathbf{r}_1)$, defined by (B.1), in eq.(2.29) we find

$$\vec{\mathcal{F}}(\mathbf{r}|\mathbf{r}_1) = \begin{cases} 0 \qquad\qquad \text{for} \quad a \leq |\mathbf{r} - \mathbf{r}_1| < 2a, \\ \\ \vec{\mathcal{F}}(\mathbf{r}|\mathbf{r}_1) \quad \text{for} \qquad |\mathbf{r} - \mathbf{r}_1| \geq 2a. \end{cases} \tag{B.2}$$

This distribution of the additional force $\vec{\mathcal{F}}(\mathbf{r}|\mathbf{r}_1)$ is represented by a function Q_{S}:

$$Q_{\mathrm{S}}(R) = \begin{cases} 0 \quad \text{for} \quad 1 \leq R < 2, \\ \\ 1 \quad \text{for} \qquad R \geq 2a, \end{cases} \tag{B.3}$$

where $R = |\mathbf{r} - \mathbf{r}_1|/a$.

Here we use the same coordinate frame and notations as in chapter 3. The fundamental equations are (2.39)-(2.44) and the boundary conditions are (3.2) and (3.3). In consequence of the definition of Q_{S}, the equation of motion becomes the Stokes type in region $1 \leq R < 2$ and the Brinkman type in $R \geq 2$. Then we also propose the solution of them as follows:

$$\begin{cases} < \mathbf{U}(\mathbf{R}|\mathbf{O}) >= U_0 \Big\{ \phi_\pm \cos\theta \mathbf{e}_r - \Big[\phi_\pm + \dfrac{R}{2} \dot{\phi}_\pm \Big] \sin\theta \mathbf{e}_\theta \Big\}, \\ \\ < Hp(\mathbf{R}|\mathbf{O}) >= -\mu U_0 \dfrac{B^2}{a} \chi_\pm \cos\theta, \end{cases} \tag{B.4}$$

where ϕ_- and χ_- are for $1 \leq R < 2$ and ϕ_+ and χ_+ are for $R \geq 2$. Substituting (B.4) into the equations of motion, respectively, we find

$$\begin{cases} \ddot{\phi}_- + \dfrac{4}{R}\dot{\phi}_- + B^2\dot{\chi}_- = 0, \\ \\ \ddot{\chi}_- + \dfrac{2}{R}\dot{\chi}_- - \dfrac{2}{R^2}\chi_- = 0, \end{cases} \quad \text{for} \quad 1 \leq R < 2, \tag{B.5}$$

$$\begin{cases} \ddot{\phi}_+ + \dfrac{4}{R}\dot{\phi}_+ + B^2\dot{\chi}_+ - B^2\phi_+ = 0, \\ \\ \ddot{\chi}_+ + \dfrac{2}{R}\dot{\chi}_+ - \dfrac{2}{R^2}\chi_+ = 0. \end{cases} \quad \text{for} \quad R \geq 2, \tag{B.6}$$

The boundary conditions are

$$\phi_-(1) = \dot{\phi}_-(1) = 0 \quad \text{and} \quad \phi_+ \to 1 \quad \text{as} \quad R \to \infty \tag{B.7}$$

Solutions of (B.5) and (B.6) are given by

$$\begin{cases} \phi_- = \dfrac{1 - 3R^2 + 2R^3}{3R^3} B^2\bar{S} + \dfrac{10R^3 - 6R^5 - 4}{60R^3} B^2\bar{T}, \\ \\ \chi_- = \bar{T}R + \dfrac{\bar{S}}{R^2}, \\ \\ \phi_+ = 1 - \dfrac{2\bar{C}_1}{R^3} + \dfrac{2\bar{C}_2}{R^3}(1 + BR)\mathrm{e}^{-BR}, \\ \\ \chi_+ = R + \dfrac{\bar{C}_1}{R^2}, \end{cases} \tag{B.8}$$

where $\bar{T}, \bar{S}, \bar{C}_1$ and $\bar{C}_2$ are unknown constants. These constants are determined by the continuity of the average volume flows of the fluid and by the continuity of the average stress tensor at $R = 2$:

$$\phi_- = \phi_+, \quad \dot{\phi}_- = \dot{\phi}_+, \quad \chi_- = \chi_+ \quad \text{and} \quad \ddot{\phi}_- = \ddot{\phi}_+.$$

By these conditions we obtain

$$B^2\bar{S} = \frac{2\left(72 + 144B + \dfrac{631}{5}B^2 + \dfrac{302}{5}B^3\right)}{96 + 96B + \dfrac{159}{15}B^2 + \dfrac{34}{15}B^3} \tag{B.9}$$

As the drag on a sphere is given by eq.(3.15) we have the self-consistency condition:

$$6c\frac{72 + 144B + \dfrac{631}{5}B^2 + \dfrac{302}{5}B^3}{96 + 96B + \dfrac{159}{15}B^2 + \dfrac{34}{15}B^3} = B^2. \tag{B.10}$$

It follows that the equation of degree five for B is obtained. Root of the algebraic equation is extracted numerically for each value of c. The non-dimensional quantity γ is shown in Fig.3.1.

Appendix C

Brinkman's method to the two-dimensional case

In this appendix we show a calculation of the non-dimensional quantity γ of the array of rods for two cases when the intermediate layer is : we set $Q_{\mathrm{C}} = 1$ and $\dot{Q}_{\mathrm{C}} = 0$.

i) For the longitudinal flow

From eq.(4.10), the basic equation which governs ψ is given by

$$\ddot{\psi} + \frac{1}{R}\dot{\psi} + B_{\mathrm{L}}^2 - B_{\mathrm{L}}^2\psi = 0. \tag{C.1}$$

Boundary conditions are $\psi(1) = 0$ and $\psi \to 1$ as $R \to \infty$. Solutions of eq.(C.1) are $\psi_0 = 1$ and $\psi_1 = \tilde{C}_0 K_0(B_{\mathrm{L}}R)$, where $K_0(r)$ is the modified Bessel function, and $\tilde{C}_0$ is the arbitrary constant. We obtain the solution which satisfies the boundary conditions:

$$\psi(R) = \psi_0 + \psi_1 = 1 - \frac{K_0(B_{\mathrm{L}}R)}{K_0(B_{\mathrm{L}})} \tag{C.2}$$

From eq.(4.14) we find the drag on the rod is given by

$$\mathbf{f} = 2\pi\mu B_{\mathrm{L}}\frac{K_1(B_{\mathrm{L}})}{K_0(B_{\mathrm{L}})}\mathbf{U}_0. \tag{C.3}$$

Hence the self-consistency condition becomes

$$B_{\mathrm{L}} = 2c\frac{K_1(B_{\mathrm{L}})}{K_0(B_{\mathrm{L}})}. \tag{C.4}$$

If we use the following expression for K_0,

$$K_0(B_{\mathrm{L}}) = \int_0^\infty \frac{\cos(B_{\mathrm{L}}t)}{\sqrt{t^2+1}}\mathrm{d}t,$$

and K_1([32],p.188),

$$K_1(B_{\rm L}) = \frac{1}{B_{\rm L}} \int_0^\infty \frac{\cos(B_{\rm L}\sinh t)}{\cosh^2 t} {\rm d}t, \tag{C.5}$$

then we get

$$\gamma = \sqrt{\frac{B_{\rm L}^2}{1-c}} = \sqrt{\frac{2c}{(1-c)} \frac{\displaystyle\int_0^\infty \frac{\cos(B_{\rm L}\sinh t)}{\cosh^2 t}{\rm d}t}{\displaystyle\int_0^\infty \frac{\cos(B_{\rm L} t)}{\sqrt{t^2+1}}{\rm d}t}}, \tag{C.6}$$

ii) For the transverse flow

In this case, from eqs.(4.24) and (4.25) we have

$$\ddot{\Psi} + \frac{3}{R}\dot{\Psi} + B_{\rm T}^2\dot{\Omega} - B_{\rm T}^2\Psi = 0, \tag{C.7}$$

$$\ddot{\Omega} + \frac{1}{R}\dot{\Omega} - \frac{1}{R^2}\Omega = 0. \tag{C.8}$$

Boundary conditions are $\Psi(1) = \dot{\Psi}(1) = 0$ and $\Psi \to 1$ as $R \to \infty$. The solution of eq.(C.8) is given by $\Omega = \tilde{C}_1 R + \tilde{C}_2/R$, where $\tilde{C}_1$ and $\tilde{C}_2$ are the unknown constants. Putting $\dot{\Omega}$ in eq.(C.7) then we find a solution Ψ_0, of it:$\Psi_0 = \tilde{C}_1 - \tilde{C}_2/R^2$. Now let $\Psi = \Psi_0 + \Psi_1$, then Ψ_1 must satisfy

$$\ddot{\Psi}_1 + \frac{3}{R}\dot{\Psi}_1 - B_{\rm T}^2\Psi_1 = 0. \tag{C.9}$$

Further, let $\Psi_1 = y(R)/R$ then eq.(C.9) becomes

$$\ddot{y} + \frac{1}{R}\dot{y} - \left(B_{\rm T}^2 + \frac{1}{R^2}\right) y = 0. \tag{C.10}$$

The solution of eq.(C.10) is also given by the modified Bessel function: $y = \tilde{C}_3 K_1(B_{\rm T} R)$ where $\tilde{C}_3$ is the unknown constant. Thus we obtain the solution of eq.(C.7) which satisfies the boundary conditions[1]

$$\Psi = \tilde{C}_1 - \frac{\tilde{C}_2}{R^2} + \frac{\tilde{C}_3}{R} K_1(B_{\rm T} R) \tag{C.11}$$

where

$$\tilde{C}_1 = 1, \quad \tilde{C}_2 = 1 + \frac{2K_1(B_{\rm T})}{B_{\rm T} K_0(B_{\rm T})}, \quad \text{and} \quad \tilde{C}_3 = \frac{2}{B_{\rm T} K_0(B_{\rm T})}.$$

[1] We use formulas for the modified Bessel function: (i)$zK_\nu'(z) + \nu K_\nu(z) = -zK_{\nu-1}(z)$, , ((ii)$K_0'(z) = -K_1(z)$).

Calculation in detail

$$\Psi(1) = 1 - \tilde{C}_2 + \tilde{C}_3 K_1(B_\mathrm{T}) = 0$$
$$\dot{\Psi}(1) = 2\tilde{C}_2 - \left\{ K_1(B_\mathrm{T}) + \frac{B_\mathrm{T}}{2}(K_0(B_\mathrm{T}) + K_1(B_\mathrm{T})) \right\} \tilde{C}_3 = 0$$

The drag on a test rod is expressed as follows:

$$\mathbf{f} = 4\pi\mu\mathbf{U}_0 \left(B_\mathrm{T}\frac{K_1}{K_0} + \frac{1}{2}B_\mathrm{T}^2 \right).$$

Therefore, from the self-consistency condition

$$B_\mathrm{T} = \frac{4cK_1}{(1-2c)K_0} \tag{C.12}$$

If we use the expression (C.5), then we have

$$\gamma = \sqrt{\frac{4c}{(1-c)(1-2c)} \frac{\displaystyle\int_0^\infty \frac{\cos(B_\mathrm{T}\sinh t)}{\cosh^2 t}\mathrm{d}t}{\displaystyle\int_0^\infty \frac{\cos(B_\mathrm{T} t)}{\sqrt{t^2+1}}\mathrm{d}t}}, \tag{C.13}$$

It should be noted that eq.(C.13) implies the divergence of B_T at $c = 1/2$. (cf. For spheres B diverges at $c = 2/3$ when the intermediate layer is neglected.) From eqs.(C.6) and (C.13) the values of γ are calculated numerically and shown in Fig.4.2. However it is difficult to integrate the formula eqs.(C.6) and (C.13) numerically, so that here we use the following asymptotic expansion for the modified Bessel function [2] [33]:

$$K_\nu(z) \simeq \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z} \left\{ 1 + \frac{4\nu^2 - 1}{8z} + \frac{(\nu^2-1)(\nu^2-9)}{2!(8z)^2} \right.$$
$$\left. + \frac{(\nu^2-1)(\nu^2-9)(\nu^2-25)}{3!(8z)^3} + \cdots \right\} \tag{C.14}$$

Therefore, for $K_0(z)$ we take into consideration to the third order

$$K_0(z) \simeq \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z} \left\{ 1 - \frac{1}{8z} + \frac{9}{2!(8z)^2} + \cdots \right\}, \tag{C.15}$$

and for $K_1(z)$

$$K_1(z) \simeq \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z} \left(1 + \frac{3}{8z} \right). \tag{C.16}$$

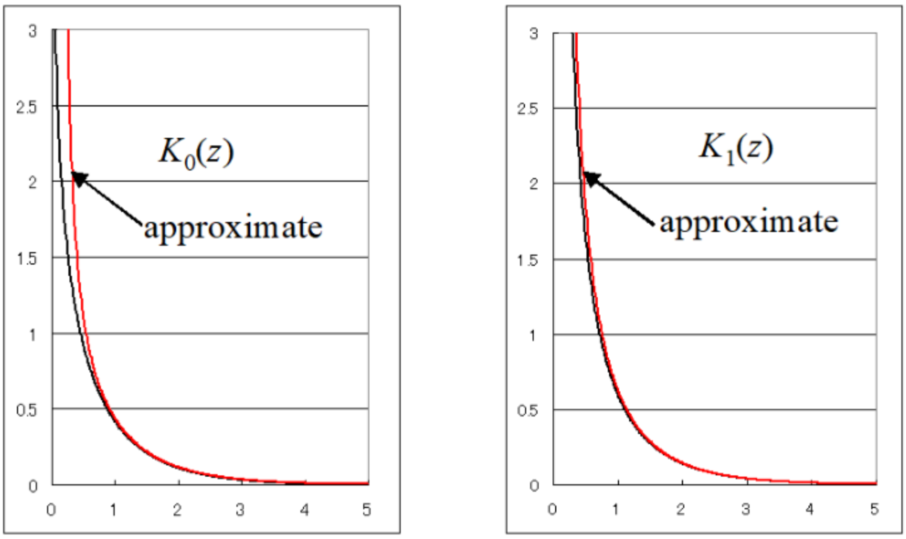

Figure C.1: Comparison of Asymtotic expansion of modified Bessel function (approximate) and exact one.

Figure C.1 shows the comparison approximate modified Bessel function (C.15) and (C.16) and integral form (C.5).

For the longitudinal flow, the non-dimensional quantity γ is smaller than those calculated by considering the intermediate layer in chapter 4. In a sense, this is somewhat paradoxical because the introduction of the layer reduces the resistance in the mean flow. However, we may say the reason as follows: By the introduction of the layer the mean flow passes more easily around the rod, so the velocity gradient, $\dot{\psi}$, becomes large, provided the no-slip condition is satisfied on it. Therefore, from eq.(4.14) the drag increases.

[2]We found this expansion from http://dspace.wul.waseda.ac.jp/ dspace/ bitstream/ 2065/ 28483/3/Honbun-4327.pdf. See also http://takeno.iee.niit.ac.jp/ shige/ math/ lecture/ misc/ data/asympt1.pdf #search=' 新潟工科大学 竹野 Bessel 関数'

Appendix D

Accuracy of the Galerkin method

In this appendix we shall solve the Brinkman equation (3.7) approximately by using the Galerkin method and then calculate the non-dimensional quantity B_{A} by the approximate solution. Then we compare the approximate B_{A} with the exact B obtained from eqn.(3.16).

(i) The trial function ϕ_{t} is assumed such that

$$\phi_{\mathrm{t}} = 1 - \frac{5}{2R^3} + \frac{3}{2R^5} + \hat{a}_1(R-1)^2\mathrm{e}^{-2(R-1)} + \hat{a}_2(R-1)^2\mathrm{e}^{-6(R-1)}. \quad \text{(D.1)}$$

The function ϕ_{t} satisfies the boundary conditions (3.9) and (3.10). The unknown coefficients $\hat{a}_j$'s can be determined by the following expression:

$$\int_1^{R_{\mathrm{F}}} \left(\ddot{\phi} + \frac{4}{R}\dot{\phi} + B^2\dot{\chi} - B^2\phi \right) \omega_j \mathrm{d}R = 0, \quad \text{(D.2)}$$

where $\dot{\chi} = 1 - 2/(3cR^3)$. Then, as in chapter 3, we have

$$\begin{cases} \hat{a}_1 = \dfrac{1}{\hat{\Lambda}} \left[\left(1.331 - \dfrac{0.3642}{c} \right) B^4 + \left(15.09 - \dfrac{4.618}{c} \right) B^2 + 5.134 \right], \\[2ex] \hat{a}_2 = \dfrac{1}{\hat{\Lambda}} \left[\left(39.49 - \dfrac{16.31}{c} \right) B^4 + \left(197.4 - \dfrac{24.97}{c} \right) B^2 + 118.8 \right], \\[2ex] \hat{\Lambda} = B^4 + 10.59B^2 + 8.788. \end{cases} \quad \text{(D.3)}$$

Since the drag on the test sphere is given by

$$\mathbf{f} = \frac{4}{3}\pi\mu\mathbf{U}_0 \left[\left(1 + \frac{1}{3c} \right) B^2 + 15 + 2\left(\hat{a}_1 + \hat{a}_2 \right) \right],$$

we have an equation of the self-consistency condition. Then it follows by substitution of (D.3) into $\hat{a}_j$ that we get the equation of degree six for B:

$$B^6 + \frac{107.2c - 40.41}{c - 2/3}B^4 + \frac{592.5c - 65.00}{c - 2/3}B^2 + \frac{379.7c}{c - 2/3} = 0. \quad \text{(D.4)}$$

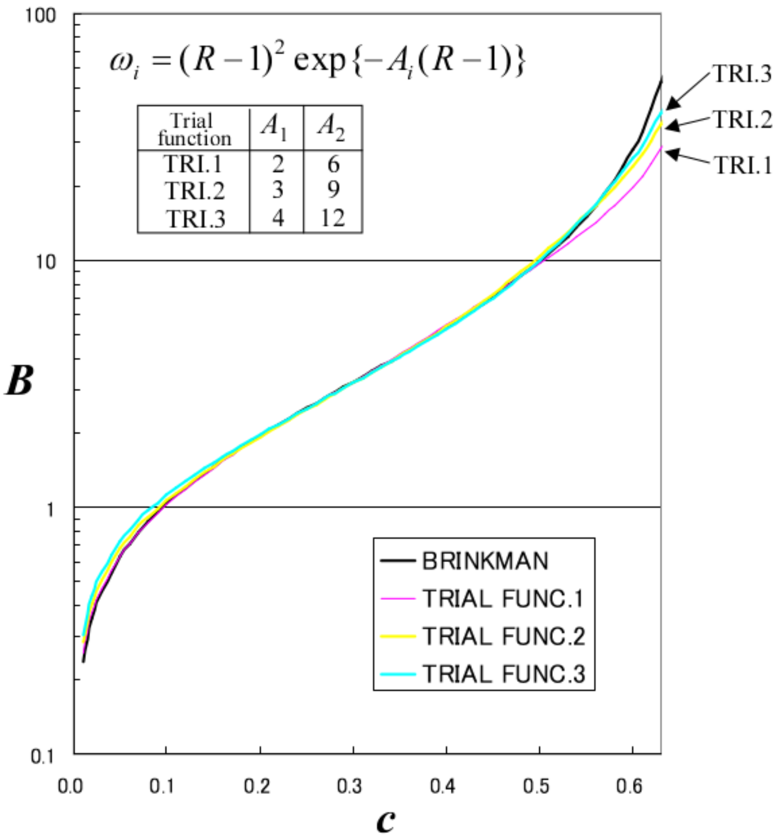

Figure D.1: Comparison of the approximate B_{A} with the exact B_{E}.

A comparison of the solution, B_{A}, of eq.(D.4) with the exact value of B_{E} of eq.(3.16) is shown in Fig.D.1. And also $\sqrt{\hat{K}_1^2/\hat{W}_1^2}$ for each concentration c is shown in Fig.D.2. Here

$$\hat{K}_1^2 = \int_1^{R_{\mathrm{F}}} \left[\ddot{\phi}_{\mathrm{t}} + \frac{4}{R}\dot{\phi} + B^2\dot{\chi} - B^2\phi_{\mathrm{t}}\right]^2 \mathrm{d}R,$$

$$\hat{W}_1^2 = \int_1^{R_{\mathrm{F}}} \left[\ddot{\phi}_{\mathrm{t}}^2 + \frac{16}{R^2}\dot{\phi}^2 + B^4\dot{\chi}^2 + B^4\phi_{\mathrm{t}}^2\right] \mathrm{d}R.$$

We can see from Fig.D.2 that the value of $\sqrt{\hat{K}_1^2/\hat{W}_1^2}$ become very small, nearly zero, around at $c = 0.3$ for all trial functions. This means that the curves represent B cross that of Brinkman around at $c = 0.3$. That is , at the cross point the trial function becomes an exact solution which is the

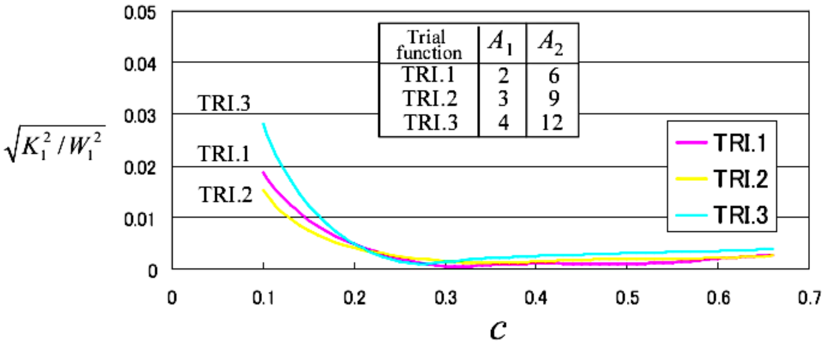

Figure D.2: The values of $\sqrt{\hat{K}_1^2/\hat{W}_1^2}$ for each concentration c.

Brinkman solution.

(ii) The trial function ϕ_t is assumed such that

$$\phi_\mathrm{t} = 1 - \frac{5}{2R^3} + \frac{3}{2R^5} + \sum_{i=1}^{2} \hat{a}_i R^2 (R-1)^2 \mathrm{e}^{-A_i(R-1)}. \qquad \text{(D.5)}$$

This form of the trial function, multiplied by R^2 to the ω_i, expresses eqn.(3.25) for the permeability of spheres introduced the intermediate layer. From Figs.D.3 and D.4 the results of accuracy calculations by several trial functions eqn.(D.5) show that the type of ω_i in (D.5) gives a sufficiently accurate function.

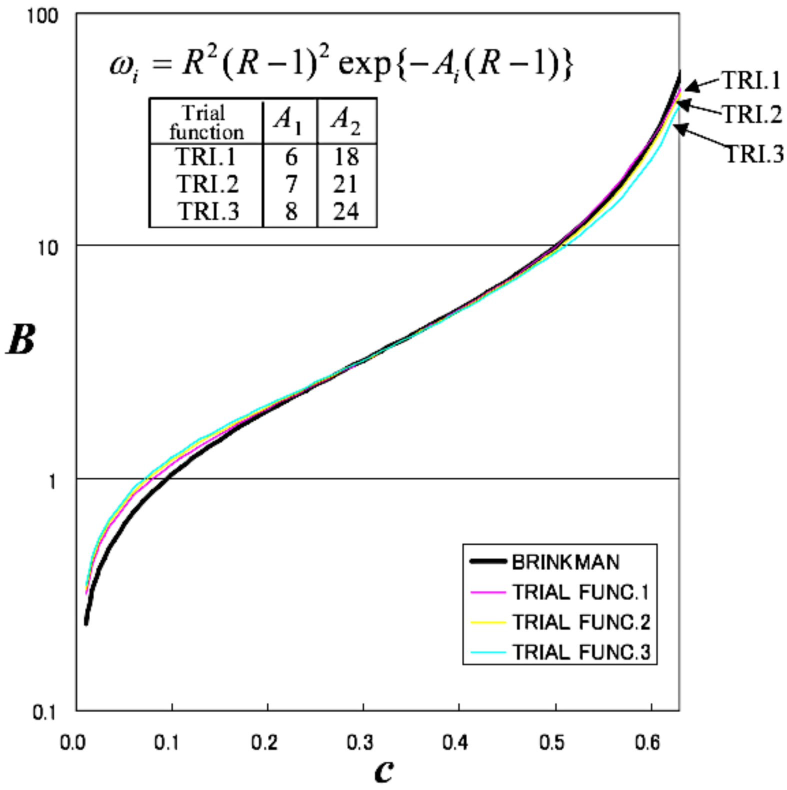

Figure D.3: Comparison of the approximate B_{A} with the exact B_{E}.

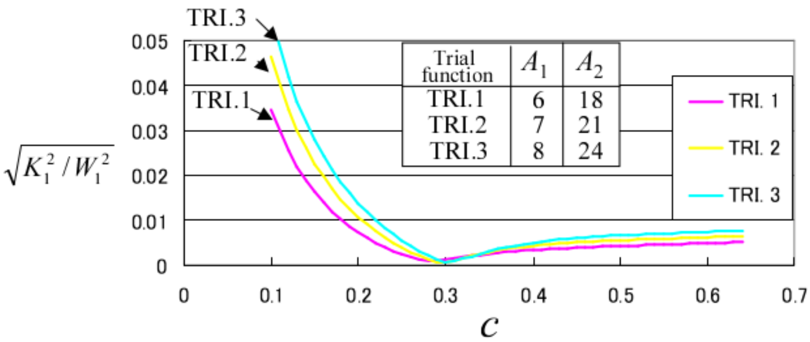

Figure D.4: The values of $\sqrt{\hat{K}_1^2/\hat{W}_1^2}$ for each concentration c.

Appendix E

Velocity profile around the test sphere

It is not so important to describe a velocity profile around the test sphere for determining the permeability of a porous medium. However it just be interested in finding a velocity profile around the sphere when considering the intermediate layer. That is to solve the simultaneous ordinary differential equations (3.21) and (3.22). To do this we use the 4-th order Runge-Kutta method.

E.1 Avoid the stiffness problem of ODE

In order to solve the fundamental equations we adopt the 4-th order Runge-Kutta scheme, however, there occurs a difficulty perhaps due to the stiffness problem of an ordinary differential equation (ODE). Usually to clear this problem one must set a step size near singular very small as possible. Here we just want to know the velocity profile approximately, so we don't adopt the usual method. We instead find a certain term which diverges in the ordinary differential equations, and then we force to converge that term along the curve which is plotted that term. Preliminary we learn how to force to converge with using Brinkman equation for spheres, which has an exact solution. The boundary condition $\Phi \longrightarrow 1$ at infinity, so that we redefine $\Phi = 1 + \phi$ and $\Pi = R + \pi$. Therefore, ϕ must be 0 at infinity. These are substituted into (3.21) and (3.22), then we get

$$\ddot{\phi} + \frac{4}{R}\dot{\phi} + B^2\left(1 + \dot{\pi}\right) - B^2 Q_{\mathrm{S}}(1 + \phi) = 0, \tag{E.1}$$

$$\ddot{\pi} + \frac{2}{R}\dot{\pi} - \frac{2}{R^2}\pi - \dot{Q}_{\mathrm{S}}(1 + \phi) = 0. \tag{E.2}$$

To conduct the Runge-Kutta scheme, the ordinary differential equations for the Brinkman equation with the intermediate layer, $Q_{\mathrm{S}}(R)$, are defined as follows:

$$\begin{cases} \dot{\phi} = \psi, \\ \dot{\psi} = -\dfrac{4}{R}\psi - B^2\chi + B^2 Q_{\rm S}(1+\phi) - B^2, \\ \dot{\pi} = \chi, \\ \dot{\chi} = -\dfrac{2}{R}\chi + \dfrac{2}{R^2}\pi - \dot{Q}_{\rm S}(1+\phi). \end{cases} \tag{E.3}$$

Initial conditions are below [cf.(3.23) and (3.27)]:

$$\phi(1) = -1, \quad \psi(1) = 0, \quad \pi(1) = \Xi - 1 + \frac{1}{3c}, \quad \chi(1) = \Xi - 1 - \frac{2}{3c}, \tag{E.4}$$

We show a procedure to avoid the divergence briefly as follows:
i) Check the term diverging in the equation $\dot{\psi}$.
ii) From Fig.E.1 we find $\dot{\psi}$ becomes to diverge (or rapidly decrease) near

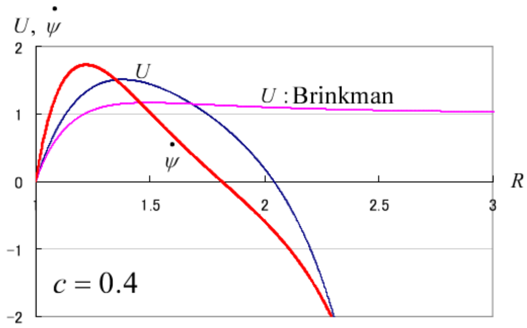

Figure E.1: Behavior of $\dot{\psi}$, a velocity U at $\theta = 90$ deg. for this case and the velocity of U_B at $\theta = 90$ deg. of the Brinkman equation.

R = 1.8 for the volume concentration c=0.4. Each term in $\dot{\psi}$ is shown in Fig.E.2.
iii) The ψ must converge to a certain value when ϕ converging to zero as R increases. That is $\dot{\psi}$ must also converge to a certain value (for simplicity here we select it zero). So that we force to modify $\dot{\psi}$ converge to zero using an exponential function which is determined by Excel optimized function research engine. The result is shown in Fig.E.3 and Fig.E.4.

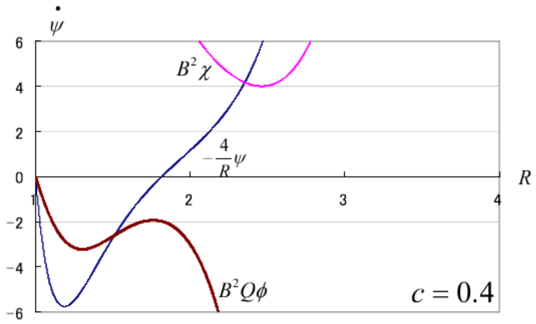

Figure E.2: Behavior of each term in $\dot{\psi}$. Each term diverges.

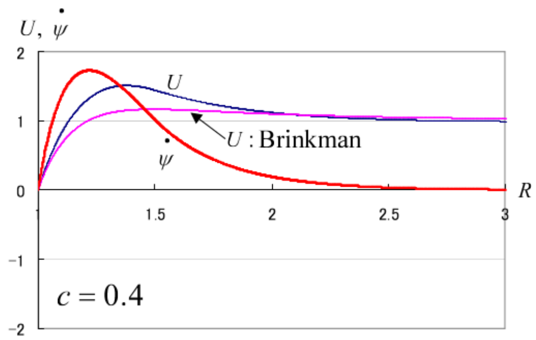

Figure E.3: Behavior of $\dot{\psi}$ and a velocity U at $\theta = 90$ deg. for this case after modification and the velocity of U_B at $\theta = 90$ deg. of the Brinkman equation.

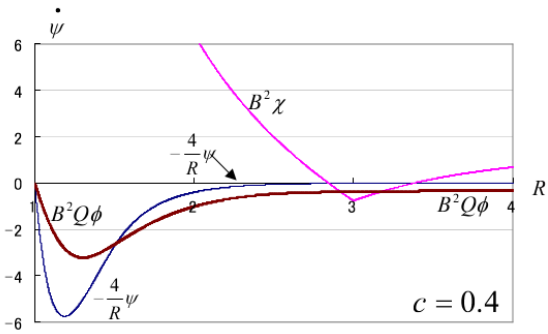

Figure E.4: Behavior of each term in $\dot{\psi}$ after modification.

E.2 Velocity profile for each volume concentration

We solve the Brinkman equation for each c by the procedure described in previous section. Velocity profile U at $\theta = 90$ degree, for instance in Fig.E.5 is shown each figure and is compared with that of Brinkman.

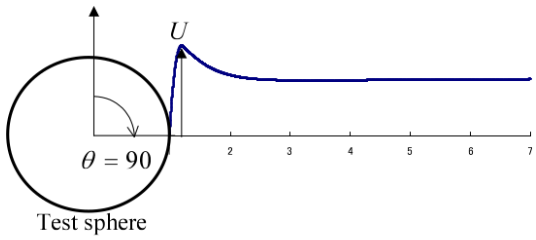

Figure E.5: Velocity profile around a test sphere.

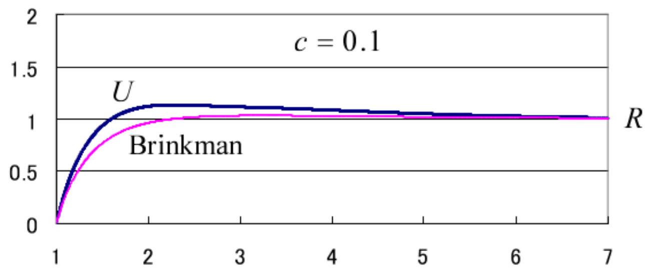

Figure E.6: Velocity profile around a test sphere.

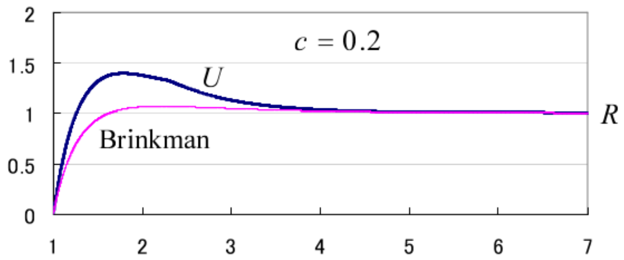

Figure E.7: Velocity profile around a test sphere.

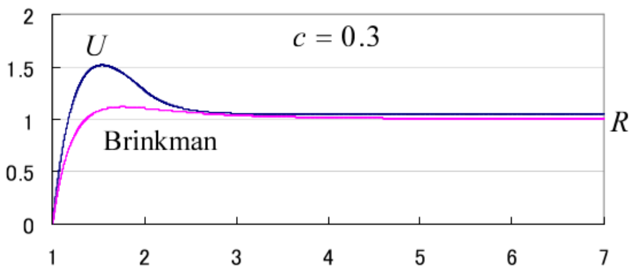

Figure E.8: Velocity profile around a test sphere.

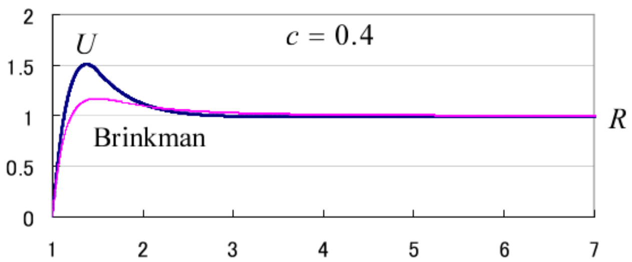

Figure E.9: Velocity profile around a test sphere.

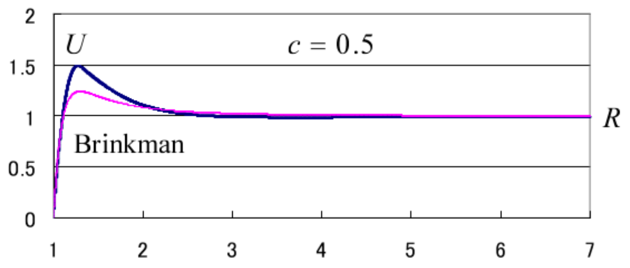

Figure E.10: Velocity profile around a test sphere.

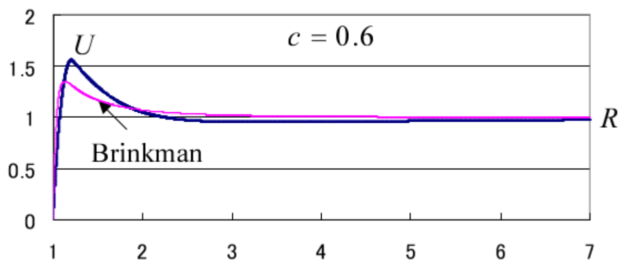

Figure E.11: Velocity profile around a test sphere.

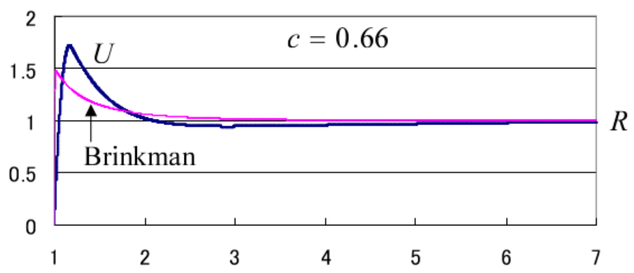

Figure E.12: Velocity profile around a test sphere.

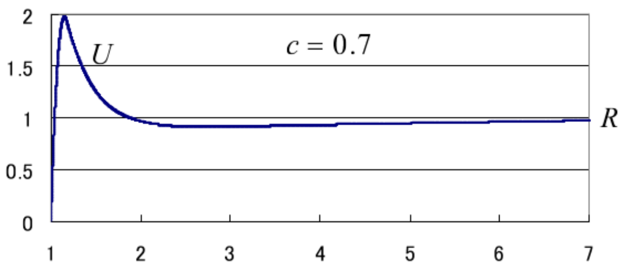

Figure E.13: Velocity profile around a test sphere.

Bibliography

[1] G. K. Batchelor : Ann. Rev. Fluid Mech. **6** (1974) 227-255

[2] A. E. Scheidegger and K. H. Liao : in “ *Fundamentals of Transport Phenomena in Porous Media*” IAHR (Elsevier Publishing Company, Amsterdam)

[3] J. Happel and H. Brenner : “*Low Reynolds Number Hydrodynamics*” (Prentice-Hall, Englewood Cliffs, N. J. , 1965)

[4] A. E. Scheidegger : “*The Physics of Flow Through Porous Media*” (MacMillan Co., New York, 1960)

[5] C. K. W. Tam : J. Fluid Mech. **38** (1969) 537-546.

[6] S. Prager : Phys. Fluid **4** (1961) 1477-1482.

[7] S. Prager : Physica **29** (1963) 129-139.

[8] G. K. Batchelor : J. Fluid Mech. **46** (1971) 813-829.

[9] G. K. Batchelor : J. Fluid Mech. **52** (1972) 245-268.

[10] Z. Hashin : Appl. Mech. Rev. **17** (1964) 1-9.

[11] R. Heczyński and I. Pieńkowska : Ann. Rev. Fluid Mech. **12** (1980) 237-269.

[12] T. S. Lundgren : J. Fluid Mech. **51** (1972) 273-299.

[13] S. Childress : J. Chem. Phys. **56** (1972) 2527-2539.

[14] I. D. Howells : J. Fluid Mech. **64** (1974) 449-475.

[15] Iu. A. Buevich and V. G. Marcov : J. Appl. Math. Mech. **36** (1972) 452-464.

[16] Iu. A. Buevich and V. G. Marcov : J. Appl. Math. Mech. **37** (1973) 1005-1021.

[17] E. J. Hinch : J. Fluid Mech. **83** (1977) 695-720.

[18] P. G. Saffman : Stud. Appl. Math. **52** (1973) 115-127.

[19] P. G.Saffman : Stud. Appl. Math. **50** (1971) 93-101.

[20] G. J. Kynch : Proc. Roy. Soc., London (A) **237** (1956) 90-116.

[21] G. H. Neal and W. K. Nader : AIChE J. **20** No.3 530-538.

[22] N. Oshima : Kikai Gakkai Koenronbun shu (1977) 97-102.

[23] J. M. De Wiest *et al.* : "*Flow through Porous Media*" (Academic Press, New York, 1969) Chap.3

[24] B. U. Felderhof: Physica **80**A (1975) 63-75.

[25] B. U. Felderhof: Physica **80**A (1975) 172-188.

[26] B. U. Felderhof & J. M. Deutch : J. Chem. Phys. **62** (1975) 2391-2397.

[27] J. M. Deutch & B. U. Felderhof: J. Chem. Phys. **62** (1975) 2398-2405.

[28] Shigenobu Itoh : J. Phys. Soc. Jpn. **52** (1983) 2379-2388.

[29] Shigenobu Itoh : J. Phys. Soc. Jpn. **53** (1984) 1718-1727.

[30] T. Kato : "*Henbunhou*" in "*Shizenkagakusya no tameno Sugaku gairon*"(Iwanami Shoten, Publishers, Tokyo, 1960)

[31] T. Hayashi & T. Mura : "*Henbunhou*" (Corona Publishing Co., Tokyo 1958).

[32] S. Moriguchi, K. Udagawa & S. Hitomatsu: "*Sugaku Koshiki III (Iwanami Zensyu 244)*" (Iwanami Shoten, Publishers, Tokyo,1960)

[33] Abramowitz, M. and Stegum, I.A., 2002, Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables, Electronic book edition," ConvertIt.com, Inc.