Derivation of Chebyshev Differentiation Matrix

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1 Lagrange interpolation polynomial

Consider the set of $u(x_j)$ for the discrete points $\{x_j\}, \quad j = 0, ..., N$. The interpolationpolynomial for $u(x_j)$in the Lagrange form is

$$p(x) = \sum_{j=0}^{N} \phi_j(x) u(x_j) \tag{1}$$

where $\phi_j(x)$ is called Lagrange interpolation coefficient and is defined by

$$\phi_j(x) = \prod_{\substack{m=0 \\ j \neq m}}^{N} \left(\frac{x - x_m}{x_j - x_m} \right). \tag{2}$$

Lagrange interpolation (1) is also defined as follows

$$\phi_j(x) = \frac{S_N(x)}{S'_N(x_j)(x - x_j)}, \tag{3}$$

where $S_N(x)$ is defined by

$$S_N(x) = \prod_{j=0}^{N} (x - x_j). \tag{4}$$

2 Chebyshev interpolation

Chebyshev polynomial is defined by

$$T_n(x) = \cos(n\theta), \quad x = \cos\theta. \tag{5}$$

For the $S_N(x)$ in the interpolation polynomial (3), we choose the following form

$$S_N(x) = (1 - x^2) \frac{\mathrm{d}T_N(x)}{\mathrm{d}x}. \tag{6}$$

By using the Chebyshev polynomial (5), we can expand $S_N(x)$ as follows:

$$S_N(x) = (1-x^2)\frac{\mathrm{d}T_N(x)}{\mathrm{d}x} = \sin^2\theta \frac{-N\sin n\theta}{-\sin\theta} = N\sin\theta\sin N\theta, \tag{7}$$

so that $S_N(x)$ becomes zero for

$$N\theta = \pi j, \quad j : \text{integer}.$$

Therefore we define

$$\theta_j = \frac{\pi j}{N}, \quad j = 0, 1, \cdots, N, \tag{8}$$

then we get

$$S_N(x_j) = 0, \quad x_j = \cos\frac{\pi j}{N}.$$

Here x_j are the Gauss-Chebyshev-Lobatto points. Thus, using (6) and Chebyshev polynomial (5), interpolation coefficient (3) is defined by

$$\phi_j(x) = \frac{(1-x^2)\frac{\mathrm{d}T_N(x)}{\mathrm{d}x}}{d_j(x-x_j)}, \quad j = 0, 1, \cdots, N, \tag{9}$$

where d_j are

$$d_j = S'_N(x_j) = -c_j N^2 T_N(x_j), \quad ' \equiv \frac{\mathrm{d}}{\mathrm{d}x} \tag{10}$$

and

$$c_j = 2, \quad \text{for} \quad j = 0, N, \quad c_j = 1 \quad \text{for} \quad 0 < j < N. \tag{11}$$

(Proof) Derivative of $S_N(x)$ with x is

$$S'_N(x) = -N\frac{\cos\theta}{\sin\theta}\sin N\theta - N^2\cos N\theta$$

Here for $j = 1, \cdots, N-1$, $\sin\theta_j \neq 0$ and $\sin N\theta_j = 0$, so that we obtain $S'_N(x_j) = -N^2 T_N(x_j)$. While $\sin\theta_j = 0$ and $\sin N \times \theta_j = 0$ for $j = 0, N$. Therefore S'_N includes $0/0$, so that we apply the L'Hopital theorem for the first term of the right hand side

$$S'_N(x_j) = S'_N(\theta = 0 \ \text{ or } \ \pi) = -N\frac{\cos\theta}{\sin\theta}\sin N\theta - N^2\cos N\theta$$

$$\overset{\text{L'Hopital}}{=} -N\frac{-\sin\theta\sin N\theta + N\cos\theta\cos N\theta}{\cos\theta} - N^2\cos N\theta = -2N^2 T_N(x_j)$$

3 Derivation of Chebyshev differentiation matrix

Consider Gauss-Chebyshev-Lobatto points (or Chebyshev points, for short) in the $x \in [-1, 1]$ defined by

$$x_j = \cos\theta_j = \frac{\pi j}{N}, \quad j = 0, 1, \cdots, N. \tag{12}$$

Given a grid function u defined on the Chebyshev points, we obtain a discrete derivative w in two steps:

• Let p be the unique polynomial of degree $\leq N$ with $p(x_j) = u_j, 0 \leq j \leq N$.
• Set $w_j = p'(x_j)$.

This operation is linear, so it can be represented by multiplication by an $(N+1) \times (N+1)$ matrix, which we shall denote by D_N:

$$w_i = (D_N)_{ij}\psi_j \tag{13}$$

Here N is an arbitrary positive integer, even or odd. And, $(D_N)_{ij}$ represents the (i, j) elements of the matrix D_N.

In order to derive the matrix D_N, consider the interpolation polynomial (1) and the interpolation coefficients (9). From the derivative $\mathrm{d}p(x)/\mathrm{d}x = u(x_j)\mathrm{d}\phi_j(x)/\mathrm{d}x$ we get

$$(D_N)_{ij} = \frac{1}{d_j}\left[\frac{\mathrm{d}}{\mathrm{d}x}\phi_j(x)\right]_{x=x_i}. \tag{14}$$

Considering eqn. (9) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi_j(x) = \frac{1}{(x-x_j)^2}\left\{\left[-2x\frac{\mathrm{d}}{\mathrm{d}x}T_N(x) + (1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}T_N(x)\right](x-x_j) - (1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}T_N(x)\right\}, \quad j = 0, 1, \cdots, N. \tag{15}$$

(I) Non-diagonal elements : $1 < i < N, \;\; 1 < j < N, \;\; i \neq j$
if we put $x = x_i$ in eqn.(15), we obtain

$$(D_N)_{ij} = \frac{1}{d_j}\frac{1}{x_i - x_j}(1-x_i^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}T_N(x_i), \tag{16}$$

where we used the relation $-2x_i\mathrm{d}T_N(x_i)/\mathrm{d}x = 0$ in the bracket [ ]. That is $\theta_i = i\pi/N, \;\; \sin\theta_i \neq 0$

$$-2x\frac{\mathrm{d}}{\mathrm{d}x}T_N(x)\bigg|_{x=x_i} = -2\cos\theta_i\frac{-N\sin i\pi}{-\sin\theta_i} = 0$$

Similarly this relation was applied for the second term of the ordinary differential equation of Chebyshev polynomial

$$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}T_n(x) - x\frac{\mathrm{d}}{\mathrm{d}x}T_n(x) + n^2 T_n(x) = 0, \tag{17}$$

then we get the following relation:

$$(1-x_i^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}T_N(x_i) = -N^2 T_N(x_i). \tag{18}$$

Substituting this into eqn.(16) we obtain

$$(D_N)_{ij} = \frac{d_i}{d_j}\frac{1}{x_i - x_j} = \frac{c_i}{c_j}\frac{T_N(x_i)}{T_N(x_j)}\frac{1}{x_i - x_j}. \tag{19}$$

Considering $T_N(x_i) = \cos[N(i\pi)/N] = \cos i\pi = (-1)^i$ and (11), we get

$$(D_N)_{ij} = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i - x_j}, \qquad c_i = \begin{cases} 2 & \text{for } i = 0 \text{ or } N \\ 1 & \text{otherwise} \end{cases} \tag{20}$$

(II) Diagonal elements $(i = j)$ $0 < i, j < N$
When $x \to x_j$, the first derivative (15)

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\phi_j(x)\right|_{x\to x_j} = \frac{1}{(x-x_j)^2}\left.\left\{\Big[Y(x)\Big](x-x_j) - (1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}T_N(x)\right\}\right|_{x\to x_j}$$

becomes 0/0 because $(1-x_j^2)\mathrm{d}T_N(x_j)/\mathrm{d}x = 0$. Where, we defined $Y(x) = -2x\mathrm{d}T_N(x_j)/\mathrm{d}x + (1-x^2)\mathrm{d}^2T_N(x_j)/\mathrm{d}x^2$, while $x \to x_j$, $Y(x_j) \neq 0$. We can apply the L'Hopital theorem to the above derivative and get

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}x}\phi_j(x)\right|_{x\to x_j} &= \frac{1}{2(x-x_j)}\Big\{ -2xT'_N + (1-x^2)T''_N \\ &+(x-x_j)\left[-2T'_N - 2xT''_N - 2xT''_N + (1-x^2)T'''_N\right] + 2xT'_N - (1-x^2)T''_N \Big\} \\ &= \frac{1}{2(x-x_j)}\Big\{(x-x_j)\Big[-2T'_N - 4xT''_N + (1-x^2)T'''_N\Big]\Big\} \\ &= \frac{1}{2}\Big\{-2T'_N - 4xT''_N + (1-x^2)T'''_N\Big\},\end{aligned} \tag{21}$$

so that the diagonal elements of the Chebyshev differentiation matrix are

$$(D_N)_{jj} = -\frac{1}{2d_j}\Big\{2T'_N(x_j) + 4x_jT''_N(x_j) - (1-x_j^2)T'''_N(x_j)\Big\}. \tag{22}$$

Because of $j \neq 0, N$, $\sin\theta_j \neq 0$, then $T'_N(x_j) = 0$. And from eqn. (17)

$$-(1-x^2)T'''_N(x) = -3xT''_N(x) + (N^2-1)T'_N(x).$$

Considering $T'_N(x_j) = 0$, we substitute this relation into eqn. (22) we obtain

$$(D_N)_{jj} = -\frac{1}{2d_j} x_j T''_N(x_j).$$

Moreover for $T''_N(x_j)$ and making use of (18), the diagonal elements are derived as follows:

$$(D_N)_{jj} = -\frac{-x_j N^2 T_N(x_j)}{2d_j(1-x_j^2)} = \frac{-x_j}{2(1-x_j^2)}. \tag{23}$$

(III) $i = 0, j = 0$ or $i = N, j = N$
Because $x_j = 1$ for $j = 0$ and $x_j = -1$ for $j = N$, then the last term of the right hand side of (22) vanishes. And we know

$$T_N(1) = 1, \;\; T_N(-1) = (-1)^N, \;\; T'_N(1) = N^2, \;\; T'_N(-1) = -(-1)^N N^2$$

and also

$$T''_N(1) = \frac{(N^2-1)N^2}{3}, \quad T''_N(-1) = -(-1)^N \frac{(N^2-1)N^2}{3}.$$

Therfore the numerator and denominator of eqn.(22) are

$$i = 0, j = 0: \quad 2T'_N(1) + 4T''_N(1) = \frac{2N^2(2N^2+1)}{3}, \quad 2d_0 = -4N^2 T_N(1)$$

$$i = N, j = N: \quad 2T'_N(-1) + 4T''_N(-1) = -(-1)^N \frac{2N^2(2N^2+1)}{3},$$

$$2d_0 = -4N^2 T_N(-1),$$

so that finally we get

$$(D_N)_{00} = \frac{2N^2+1}{6}, \qquad (D_N)_{NN} = -\frac{2N^2+1}{6}. \tag{24}$$

Derivation of $T''_N(\pm 1)$:
Ordinary differentiation equation for Chebyshev polynomial in (V) we used

$$-(1-x^2)T'''_N(x) = -3xT''_N(x) + (N^2-1)T'_N(x).$$

By using for $x = 1$ in above equation, we obtain

$$T''_N(1) = \frac{(N^2-1)}{3} T'_N(1) = \frac{(N^2-1)N^2}{3}.$$

For $T''_N(-1)$, we derive similarly.

References

[1] Trefethen, L. N.: *2000 Spectral Methods in Matlab, SIAM.*

[2] C. Canuto, A. Quarteroni, M. Y. Hussaini and T. A. Zang: *Spectral Methods Fundamentals in Single Domains*, Springer-Verlag Berlin Heidelberg 2006

[3] Haydar Alici: Pseudospectral Methods for Differential Equations: Application to the Schrodinger Type Eigenvalue Problems, December 2003, The Middle East Technical University