On the Laguerre differentiation matrix

Shigenobu Itoh

February 10, 2019

1 Interpolation polynomial

Consider the set of $u(x_j)$ for the discrete points $\{x_j\}, \;\; j=0,1,....,N$. The interpolation polynomial for $u(x_j)$in the Lagrange form is

$$p(x)=\sum_{j=0}^{N}\phi_j(x)u(x_j) \tag{1}$$

where $\phi_j(x)$ is called Lagrange interpolation coefficient and is defined by

$$\phi_j(x)=\prod_{\substack{m=0\\m\neq j}}^{N}\left(\frac{x-x_m}{x_j-x_m}\right). \tag{2}$$

Lagrange interpolation (2) is also defined as follows

$$\phi_j(x)=\frac{S_N(x)}{S'_N(x_j)(x-x_j)}, \tag{3}$$

where $S_N(x)$ is defined by

$$S_N(x)=\prod_{m=0}^{N}(x-x_m).$$

2 Laguerre polynomial $L_N(x)$ and $L_N(x_j)=0$

Consider following Laguerre polynomial :

$$L_n(x)=\sum_{r=0}^{n}(-1)^r{}_nC_{n-r}\frac{x^r}{r!}. \tag{4}$$

The ordinary differential equation (ODE, for short) for the Laguerre polynomial y

$$xy''+(1-x)y'+ny=0. \tag{5}$$

The recurrence relation for Laguerre polynomial (4) is given by as follows

$$nL_{n-1}(x) - (2n+1-x)L_n(x) + (n+1)L_{n+1}(x) = 0 \tag{6}$$

In the following sections, the roots of $L_N(x) = 0$ are needed. Therefore, a method for obtaining this value will be described. By the recurrence relation (6), the following matrix is derived

$$\begin{bmatrix} 1-x & -1 & 0 & \cdots & 0 \\ -1 & 3-x & -2 & \ddots & \vdots \\ 0 & -2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2N-1-x & -N \\ 0 & \cdots & 0 & -N & 2N+1-x \end{bmatrix} \begin{bmatrix} L_0 \\ L_1 \\ \vdots \\ \vdots \\ L_{N-1} \\ L_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ b_{N+1} \end{bmatrix} \tag{7}$$

Here, if we require $L_{N+1}(x) = 0$ then the system reduces to a standard eigenvalue problem $W\mathbf{r} = x\mathbf{r}$ with eigenvalue parameter x, which provides the roots $x_j, \;\; j = 0, 1, \cdots, N$ of $L_{N+1}(x)$ as required.

Example The roots of $L_{50}(x) = 0$
minimum root $x_1 = 0.02863051833938032$,
maximum root $x_{50} = 180.698343709214555$

3 Laguerre differentiation matrix for the first derivative

If we choose the Laguerre polynomial (4) for the interpolation coefficient $S_N(x) \equiv L_N(x)$, $S_N(x)$ does not become zero at $x = 0$, because $L_N(0) \neq 0$. Therefore we choose

$$S_N(x) = xL_N(x). \tag{8}$$

Therefore

$$S'_N(x) = L_N(x) + xL'_N(x). \tag{9}$$

and the interpolation coefficient $\phi_j(x)$ by Laguerre polynomial is defined by

$$\phi_j(x) = \frac{xL_N(x)}{S'_N(x_j)(x - x_j)}. \tag{10}$$

The first derivative of $p(x)$ defined by (1) is written by

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}p(x)\right|_{x=x_i} = \sum_{j=0}^{N} \left.\frac{\mathrm{d}}{\mathrm{d}x}\phi_j(x)\right|_{x=x_i} u(x_j) \equiv \sum_{j=0}^{N} (D_N)_{ij} u(x_j)$$

In order to derive the elements of the differentiation matrix D_N, we differentiate the $\phi_j(x)$ and get

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi_j(x) = \frac{L_N(x) + xL'_N(x)}{S'_N(x_j)(x - x_j)} - \frac{xL_N(x)}{S'_N(x_j)(x - x_j)^2}. \tag{11}$$

(i) Non-diagonal elements :$i \neq j, \;\; 0 < i, j \leq N$
From eqn.(11)

$$(D_N)_{ij} = \frac{\mathrm{d}}{\mathrm{d}x}\phi_j(x_i) = \frac{x_i L'_N(x_i)}{S'_N(x_j)(x_i - x_j)}$$

where

$$S'_N(x_j) = x_j L'_N(x_j),$$

so that we obtain

$$(D_N)_{ij} = \frac{1}{(x_i - x_j)} \frac{x_i L'_N(x_i)}{x_j L'_N(x_j)}. \tag{12}$$

(ii) Case $i = 0, \;\; 0 < j \leq N$:
We put $x_i = x_0 = 0$ for eqn.(11) and take into consideration $L_N(0) = 1$, the elements are derived as follows

$$(D_N)_{0j} = \frac{L_N(0)}{S'_N(x_j)(-x_j)} = -\frac{1}{x_j^2 L'_N(x_j)}. \tag{13}$$

(iii) Case $j = 0, \;\; 0 < i \leq N$:
By eqns.(9) and (10), the interpolation coefficient for special case becomes

$$\phi_0(x) = \frac{L_N(x)}{S'_N(0)} = L_N(x).$$

Then we get

$$(D_N)_{i0} = L'_N(x_i). \tag{14}$$

(iv) Diagonal elements $i = j$:
From eqn. (11), the derivative of $\phi_j(x)$ for the case becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi_j(x) = \frac{[L_N(x) + xL'_N(x)](x - x_j) - xL_N(x)}{S'_N(x_j)(x - x_j)^2}.$$

For above equation we put $x \to x_j$, then the numerator and the denominator become zero. Therefore, we can apply the L'Hopital theorem to it:

$$\stackrel{\text{L'Hopital}}{=} \frac{[L'_N(x) + L'_N(x) + xL''_N(x)](x - x_j) + (L_N(x) + xL'_N(x)) - L_N(x) - xL'_N(x)}{2S'_N(x_j)(x - x_j)}$$

$$= \frac{xL''_N(x) + 2L'_N(x)}{2S'_N(x_j)} = \frac{xL''_N(x) + 2L'_N(x)}{2x_j L'_N(x_j)}$$

Here the following relation

$$xL''_N(x) = (x-1)L'_N(x) - NL_N(x)$$

is derived from ODE (5), then we substitute this into the most right hand side of the diagonal elements, and get

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi_j(x) = \frac{(x+1)L'_N(x) - NL_N(x)}{2x_j L'_N(x_j)}.$$

Therefore we put $x \to x_j$, the diagonal elements of the matrix are

$$(D_N)_{jj} = \frac{x_j + 1}{2x_j} \tag{15}$$

(v) Case $x = 0$ $(i = j = 0)$:
For eqn. (10), we put $x = x_j = 0$, and in eqn. (9) considering $S'_N(0) = L_N(0) = 1$ then we obtain

$$\phi_0(x) \overset{x \to 0}{=} \frac{xL_N(x)}{S'_N(0)x} = L_N(x).$$

Thus

$$(D_N)_{00} = \frac{\mathrm{d}}{\mathrm{d}x}L_N(x)\bigg|_{x=0} = L'_N(0) = -N. \tag{16}$$

This corresponds to eqn.(14) for the case (III). So, we put $x_i \to x_0 = 0$, then we get the result (16).
Numerical results 1 : We try the above differentiation matrix for the function $U(x) = \exp(-x)\sin(x)$, $N = 50$, and the results are shown in Table 1. It should be noted that when x becomes large the error also large. Then in order to avoid those inaccurate elements we set the size of the matrix $M = 14$: $(D_N) = (14 \times 14)$.

4 Laguerre differentiation matrix for the second derivative

Differentiate eqn. (11) further, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi_j(x) = \frac{2L'_N(x) + xL''_N(x)}{S'_N(x_j)(x-x_j)} - 2\frac{L_N(x) + xL'_N(x)}{S'_N(x_j)(x-x_j)^2} + 2\frac{xL_N(x)}{S'_N(x_j)(x-x_j)^3}. \tag{17}$$

Table 1: $U(x) = \exp(-x)\sin(x), \quad N = 50, M = 14$

x_j	Results by D_N	Exact values
0	1.000000005	1
0.028630518	0.943558556	0.943558559
0.150882936	0.720918497	0.720918492
0.370948782	0.392989097	0.392989103
0.689090700	0.068270825	0.068270815
1.105625024	-0.147352646	-0.147352632
1.620961751	-0.207373689	-0.207373709
2.235610376	-0.150119367	-0.150119335
2.950183367	-0.061329775	-0.061329828
3.765399774	-0.005269365	-0.005269243
4.682089388	0.008974974	0.008974876
5.701197575	0.004628096	0.004628807
6.823790910	0.000374024	0.000372763
8.051063669	-0.000377014	-0.000375009

(i) Non-diagonal elements : $i \neq j, \quad 0 < i, j \leq N$
Considering $L_N(x_i) = 0$ in eqn.(17) we get

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi_j(x)\bigg|_{x=x_i} = \frac{2L_N'(x_i) + x_i L_N''(x_i)}{S_N'(x_j)(x_i - x_j)} - \frac{2x_i L_N'(x_i)}{S_N'(x_j)(x_i - x_j)^2}$$

By using ODE (5), we find the relation $x_i L_N''(x_i) = (x_i - 1)L_N'(x_i)$, then

$$(\text{above equation}) = \frac{(x_i + 1)L_N'(x_i)}{S_N'(x_j)(x_i - x_j)} - \frac{2x_i L_N'(x_i)}{S_N'(x_j)(x_i - x_j)^2},$$

where $S_N'(x_j) = x_j L_N'(x_j)$. Therefore, we obtain the following expression of non-diagonal elements

$$(D_N^{(2)})_{ij} = \frac{[(x_i + 1)(x_i - x_j) - 2x_i]L_N'(x_i)}{x_j(x_i - x_j)^2 L_N'(x_j)}. \tag{18}$$

(ii) The first row :$i = 0, \quad 0 < j \leq N$
When deriving the result (18), the relational expression$x_i L_N''(x_i) = (x_i - 1)L_N'(x_i)$ was used. However, putting $x_i = 0$ for the relational expression, then incorrect result yields:$0 = -L_N'(0) = N$. Therefore we must return to eqn. (17). And putting $x = x_i = 0$ in this equation, we find

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi_j(0) = -\frac{2L_N'(0)}{S_N'(x_j)x_j} - \frac{2L_N(0)}{S_N'(x_j)x_j^2}$$

that is

$$(D_N^{(2)})_{0j} = \frac{2N}{x_j^2 L_N'(x_j)} - \frac{2}{x_j^3 L_N'(x_j)}. \tag{19}$$

(iii) Diagonal elements:$i = j, \ \ 0 < i, j \leq N$
We can rewrite eqn.(17) as follows

$$\phi_j''(x) = \frac{(2L_N'(x) + xL_N''(x))(x - x_j)^2 - 2(L_N(x) + xL_N'(x))(x - x_j) + 2xL_N(x)}{S_N'(x_j)(x - x_j)^3}.$$

When $x \to x_j$, the numerator and denominator become zero. So, applying the L'Hopital theorem, differentiate the numerator and denominator. First the denominator becomes

$$3S_N'(x_j)(x - x_j)^2,$$

the second the numerator becomes

$$(3L_N'' + xL_N''')(x - x_j)^2 + 2(2L_N' + xL_N'')(x - x_j) - 2(2L_N' + xL_N'')(x - x_j)$$

$$-2(2L_N' + xL_N'')(x - x_j) - 2(L_N + xL_N') + 2L_N + 2xL_N' = (3L_N'' + xL_N''')(x - x_j)^2.$$

From these results we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi_j(x) = \frac{3L_N''(x) + xL_N'''(x)}{3S_N'(x_j)}. \tag{20}$$

By ODE(5)

$$xy''' + (2 - x)y'' + (n - 1)y' = 0 \Longrightarrow xL_N''' = (x - 2)L_N'' + (1 - N)L_N'.$$

We substitute the obtained relation into the numerator of the right hand side of (20), then obtain

$$3L_N'' + \{(x-2)L_N'' + (1-N)L_N'\} = (x+1)L_N'' + (1-N)L_N' = xL_N'' + L_N'' + (1-N)L_N'$$

$$= (x - 1)L_N' - NL_N + \frac{(x - 1)L_N' - NL_N}{x} + (1 - N)L_N'$$

$$= \left(x - N + \frac{x - 1}{x}\right) L_N'(x) - \left(1 + \frac{1}{x}\right) NL_N(x).$$

Thus we put $x \to x_j$ in eqn. (20), and get

$$(D_N^{(2)})_{jj} = \left(x_j - N + \frac{x_j - 1}{x_j}\right) \frac{1}{3x_j}. \tag{21}$$

(iv) The first column: $j = 0, \ \ 0 < i \leq N$

Table 2: $U(x) = \exp(-x)\sin(x), \quad N = 50, M = 14$

x_j	$D_N^{(2)}$	$D_N \times D_N$	Exact
0	-2.000000451	-2.000000493	-2
0.028630518	-1.942754490	-1.942754464	-1.942754386
0.150882936	-1.700356508	-1.700356550	-1.700356544
0.370948782	-1.286285499	-1.286285439	-1.286285474
0.689090700	-0.774961725	-0.774961810	-0.774961749
1.105625024	-0.296960809	-0.296960685	-0.296960780
1.620961751	0.019827976	0.019827788	0.019827939
2.235610376	0.131929026	0.131929324	0.131929077
2.950183367	0.102748887	0.102748384	0.102748820
3.765399774	0.037593220	0.037594131	0.037593439
4.682089388	0.000561324	0.000559532	0.000561041
5.701197575	-0.00558386	-0.005579935	-0.005583559
6.823790910	-0.001862931	-0.001872984	-0.001864993
8.051063669	0.000117415	0.000152662	0.000124833

As already used several times, putting $x_j = 0$ in eqn. (10), yields $\phi_0(x) = L_N(x)$. Therefore

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi_0(x) = L_N''(x).$$

Then $L_N''(x)$ is represented by $L_N'(x)$ from ODE (5), thus

$$(D_N^{(2)})_{i0} = \frac{x_i - 1}{x_i} L_N'(x_i). \tag{22}$$

(v) Case $i = j = 0$:
This is the case that $x = 0$ in the above expression $\mathrm{d}^2\phi_0(x)/\mathrm{d}x^2 = L_N''(x)$. We put $x = 0$ in the obtained equation $xy''' + (2-x)y'' + (n-1)y' = 0$,

$$2L_N''(0) = (1-N)L_N'(0) \Longrightarrow L_N''(0) = \frac{1-N}{2}L_N'(0) = \frac{N(N-1)}{2},$$

Thus

$$(D_N^{(2)})_{00} = \frac{N(N-1)}{2}. \tag{23}$$

Numerical results 2: Similar as Numerical results 1, comparison of the results by $D_N^{(2)}$, $D_N \times D_N$ and exact values are shown in Table 2.